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Effective Potentials and Vacuum Structure in $N = 1$ Supersymmetric Gauge Theories

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ABSTRACT

We derive the exact effective superpotential in $4d$, $N = 1$ supersymmetric $SU(2)$ gauge theories with N_A triplets and $2N_f$ doublets of matter superfields. We find the quantum vacua of these theories; the equations of motion (for $N_A = 1$) can be reorganized into the singularity conditions of an elliptic curve. From the phase transition points to the Coulomb branch, we find the exact Abelian gauge couplings, τ , for arbitrary bare masses and Yukawa couplings. We thus *derive* the result that τ is a section of an $SL(2, \mathbf{Z})$ bundle over the moduli space and over the parameters space of bare masses and Yukawa couplings. For $N_c > 2$, we

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derive the exact effective superpotential in branches of supersymmetric $SU(N_c)$ gauge theories with one supermultiplet in the adjoint representation ($N_A = 1$) and zero or one flavor ($N_f = 0, 1$). We find the quantum vacua of these theories; the equations of motion can be reorganized into the singularity conditions of a genus $N_c - 1$ hyperelliptic curve. Finally, we present the effective superpotential in the $N_A, N_f < N_c$ cases.

1 Introduction and Discussion

Recently, many new exact results were derived in four dimensional supersymmetric gauge theories (for a review, see ref. [1] and references therein). In particular, in ref. [2], we reported the results of applying the methods of refs. [1, 3, 4, 5] to the general case of an infra-red nontrivial $N = 1$ supersymmetric $SU(2)$ gauge theory with N_A triplets of matter superfields and N_f , $N = 2$ flavors (*i.e.*, $2N_f$ doublets). In [2], we presented the exact effective superpotential in these models, and the effective Abelian gauge couplings for arbitrary bare masses and Yukawa couplings.

In this paper, we present the detailed derivation of the results of ref. [2]. Moreover, we extend the results to the “ $SU(N_c)$ vacua” branches of supersymmetric $SU(N_c)$ gauge theories with N_A matter superfields in the adjoint representation, N_f supermultiplets in the fundamental and N_f supermultiplets in the anti-fundamental representations ($N_f < N_c$).

To derive the nonperturbative superpotential of a particular supersymmetric gauge theory, one may attempt to obtain a unique result by using holomorphy, symmetries and limiting considerations [1]. An equivalent, but more efficient way, in some cases, is to derive the exact superpotential by applying similar considerations in an “integrating in” procedure [3, 4]. Under certain conditions, one may, unconventionally, derive the effective superpotential for modes which are of finite mass, given the effective action in which these modes have been considered to have infinite mass. In this work, we apply the integrating in technique when it is valid; the various consistency checks to which the result is subjected strengthen the reliability of the method.

We begin, in section 2, with a review of the integrating in procedure. We discuss the limiting considerations which determine when such a procedure allows to derive the nonperturbative superpotential. Then, in section 3, we review the integrating in of $2N_f$ doublets to pure $N = 1$ supersymmetric $SU(2)$ gauge theory.

In section 4, we integrate in N_A triplets to an $N = 1$ supersymmetric $SU(2)$ gauge theory with $2N_f$ doublets. We find a universal representation

of the nonperturbative superpotential for all (one-loop) infra-red nontrivial theories, namely, with $N_A = 3$, $N_f = 0$, or $N_A = 2$, $N_f = 0, 1, 2$, or $N_A = 1$, $N_f = 0, 1, 2, 3, 4$. We then review the physics of cases without doublets ($N_f = 0$).

In sections 5,6,7,8, we consider in detail the $SU(2)$ models with one triplet matter superfield ($N_A = 1$), and with $N_f = 1, 2, 3, 4$, respectively. In all cases, we find the quantum vacua in the Higgs/confinement [6] branches (the “ $SU(2)$ vacua”). We reorganize some of the equations, derived by variations of the superpotential, into the singular conditions of an elliptic curve. This elliptic curve has $2 + N_f$ singularities, corresponding to the $2 + N_f$ branches of $SU(2)$ vacua; the values of the quantum field, corresponding classically to the $SU(2)$ quadratic Casimir, are fixed. The rest of the equations of motion determine the values of other quantum fields as functions of the bare masses and Yukawa couplings in each branch. In the massless case, there is a Z_{4-N_f} global symmetry⁵ acting on the moduli space of the $N_A = 1$, $N_f > 0$ theory; this symmetry can be read directly from the quantum superpotential.

Moreover, at the phase transition points to the Coulomb branch, the results in sections 5,6,7,8 provide a direct derivation of the elliptic curves, defining the effective Abelian gauge coupling, τ , as a function of the bare masses and Yukawa couplings. Therefore, we *derive* the result that τ is a modular parameter of a torus, namely, a section of an $SL(2, \mathbf{Z})$ bundle over the moduli space and over the parameters space of bare masses and Yukawa couplings.

These results pass various consistency checks (like integrating out of any degree of freedom). In particular, on the subspace of parameters where the theory has an enhanced $N = 2$ supersymmetry, we reobtain the results of Seiberg and Witten [7, 8]. On the way, we identify a physical meaning of the complex parameter x appearing in the elliptic curve equation: $y^2 = p(x)$.

At each of the $2 + N_f$ singular points, in the moduli space of the Coulomb branch, a dyon becomes massless. For special values of the bare masses and Yukawa couplings, some of the $2 + N_f$ vacua degenerate. In some cases, it may lead to points where mutually non-local degrees of freedom are mass-

⁵ Z_1 means no symmetry and Z_0 means $U(1)$.

less ⁶, similar to the situation in pure $N = 2$ supersymmetric $SU(3)$ gauge theories, considered in [9]. For example, when all masses and Yukawa couplings approach zero, all the $2 + N_f$ singularities collapse to the origin. Such a point might be interpreted as a non-Abelian Coulomb phase [10].

In sections 9,10, we consider the $SU(2)$ models with two triplet matter superfields ($N_A = 2$), and with $N_f = 1, 2$. In section 9, we find the quantum vacua of the model with one flavor ($N_f = 1$). Again, at the phase transition points to the Coulomb branch, the equations derived by variation of the superpotential can be reorganized into the singular conditions of an elliptic curve – the one defining the effective Abelian gauge coupling in the Coulomb branch. Unlike the $N_A = 1$ cases, away from the phase transition points, the equations determining the vacua fail to describe, in general, an elliptic curve. As before, on the way, we identify a physical meaning of the complex parameter x , which at the phase transition points becomes the one appearing in the elliptic curve equation: $y^2 = p(x)$. For special values of the bare parameters, we consider a point in the moduli space that might be interpreted as a non-abelian Coulomb phase [10] and, as another consistency check, we show that the reduction from $N_A = 2$ to $N_A = 1$ is simple.

In section 10, we argue that the supersymmetric $SU(2)$ gauge theory with $N_A = N_f = 2$ is infra-red free. This result is consistent with the fact that (unlike the other one-loop conformal cases: $N_A = 1$, $N_f = 4$, or $N_A = 3$, $N_f = 0$) we are not able to match the coupling constant of this theory to the one of the model with $N_A = 0$, $N_f = 2$, in a way that respects the global symmetries of the theory. The general discussion in section 10 follows ref. [11]. Analyzing the gauge-coupling beta-function and the Yukawa couplings beta-functions, we see that in the other one-loop infra-red conformal theories, indeed, the beta-function equations are expected to have a fixed line of solutions.

In sections 11,12,13, we turn to supersymmetric $SU(N_c)$ gauge theories with more than two colors: $N_c > 2$. In section 11, we consider $SU(N_c)$ with one matter superfield in the adjoint representation ($N_A = 1$) and no flavors ($N_f = 0$). In this case, symmetries and limiting considerations are

⁶We thank R. Plesser and N. Seiberg for discussions on this point.

not strong enough to allow a determined integrating in procedure from the pure supersymmetric $SU(N_c)$ gauge theory (although they are enough to show that the effective superpotential vanishes). However, imposing also the physical condition to have a discrete set of “ $SU(N_c)$ vacua,” allows one to obtain the result: the nonperturbative superpotential vanishes and there are constraints which fix the quantum field, corresponding classically to the $SU(N_c)$ quadratic Casimir, at N_c values, and fix the value of quantum fields, corresponding classically to higher Casimirs, to zero. These are exactly the N_c $SU(N_c)$ vacua of the theory; there is a Z_{N_c} global symmetry, acting in the moduli space, which relates them.

In section 12, we present the nonperturbative superpotential in supersymmetric $SU(N_c)$ gauge theory with N_f flavors and $N_A = 0$, found in refs. [12, 13]. Then, in section 13, we add to the $N_f = 1$ theory a matter superfield in the adjoint representation. As before, symmetries and limiting considerations are not strong enough to allow a determined integrating in procedure from the supersymmetric $SU(N_c)$ gauge theory with $N_f = 1$ to the theory with $N_A = N_f = 1$. However, imposing also the physical condition to have a discrete set of $SU(N_c)$ vacua, allows one to obtain the result: the nonperturbative superpotential does not depend on the quantum fields, corresponding classically to $SU(N_c)$ Casimirs, except for the quadratic Casimir; the quantum fields corresponding to higher Casimirs are constrained.

We find the quantum vacua in the fully Higgsed/confined branches (the “ $SU(N_c)$ vacua”) in the presence of a tree-level superpotential containing only mass terms and a Yukawa coupling term. We reorganize some of the equations, derived by variations of the superpotential, into the singular conditions of a genus $N_c - 1$ hyperelliptic curve. This hyperelliptic curve has $2N_c - 1$ singularities, corresponding to the $N_c + (N_c - 1)N_f = 2N_c - 1$ $SU(N_c)$ vacua of the $N_A = N_f = 1$ theory. In the massless case, there is a $Z_{2N_c - 1}$ global symmetry acting on the moduli space of this theory; this symmetry can be read directly from the quantum superpotential.

In section 14, we revisit the supersymmetric $SU(2)$ gauge theory with $N_A = N_f = 1$, in a way similar to the manipulation for $N_c > 2$. We find that the equations of motion, determining the $SU(2)$ vacua, are the singularity

conditions of an elliptic curve, related to the previous one by an $SL(2, \mathbf{C})$ transformation.

In section 15, we present the effective superpotential in supersymmetric $SU(N_c)$ gauge theories, $N_c > 2$, with N_A matter superfields in the adjoint representation and $N_f < N_c$ flavors. For $N_A = 1$, we find that there are $N_c + N_f(N_c - 1) - \frac{1}{2}N_f(N_f - 1)$ (branches of) $SU(N_c)$ vacua in the presence of a tree-level superpotential containing only mass terms and Yukawa coupling terms. In the massless case, there is a $Z_{2N_c - N_f}$ global symmetry acting on the moduli space of the $N_A = 1$, $N_f > 0$ theory; this symmetry can be read directly from the quantum superpotential. For $N_A = 2$, $N_f \neq 0$, the superpotential in section 15 is conjectured.

Finally, in the Appendix, we show in detail the considerations leading to the conclusion that the integrating in procedure is valid in examples discussed in this work.

2 Integrating in

For completeness, we start by repeating the general discussion of refs. [3, 4]. Let the nonperturbative superpotential of an $N = 1$ supersymmetric gauge theory, which we call the “down” theory, be

$$W_d \equiv W_{down}(X_I, \Lambda_d). \quad (2.1)$$

W_d depends on the gauge singlets X_I , which are constructed out of the down theory matter superfields, D_i , and on the gauge coupling constant, expressed as the dynamically generated scale of the down theory, Λ_d ; we may add to W_d the tree-level superpotential: $m^I X_I$.

Suppose we also know the nonperturbative superpotential of the $N = 1$ gauge theory with a matter superfield U in addition to D_i . We call this theory the “up” theory, and its nonperturbative superpotential is

$$W_u \equiv W_{up}(X_I, M, Z_A, \Lambda), \quad (2.2)$$

where $M \sim U^2$ and Z_A are relevant gauge singlets constructed out of U and D_i . In eq. (2.2), $\Lambda \equiv \Lambda_u$ is the dynamically generated scale of the up theory.

One may add to W_u the tree-level superpotential

$$W_{tree} = \tilde{m}M + \lambda^A Z_A. \quad (2.3)$$

Here \tilde{m} is the mass of the superfield U , and λ^A are some couplings of U and D_i .

By integrating out U at finite mass, \tilde{m} , one gets, using the notation of ref. [4],

$$[W_u + \tilde{m}M + \lambda Z]_{\langle M \rangle, \langle Z \rangle} = W_d + W_i, \quad (2.4)$$

where

$$W_i \equiv W_{intermediate}(X, \tilde{m}, \lambda). \quad (2.5)$$

We may now split the “intermediate” superpotential into two pieces:

$$W_i = W_{tree,d} + W_\Delta, \quad (2.6)$$

where the tree-level down superpotential, $W_{tree,d}$, is

$$W_{tree,d} \equiv W_{tree}|_{\langle U \rangle}. \quad (2.7)$$

So far we have described the obvious “integrating out” of U from the up theory to the down theory. Now, suppose we start with the down theory, and add to it the superfield U . One may attempt to obtain a unique result for W_u by symmetry and limiting considerations. A more efficient way, in some cases, is to derive W_u by applying similar considerations in an “integrating in” procedure.

The integrating in of U from the down theory to the up theory is possible if we know W_Δ .⁷ The superpotential of the up theory is derived from the superpotential of the down theory by the Legendre transform of eq. (2.4):

$$W_u = [W_d + W_\Delta + W_{tree,d} - W_{tree}]_{\langle \tilde{m} \rangle, \langle \lambda \rangle}. \quad (2.8)$$

How can one find W_Δ ? It might happen that holomorphy, symmetries, and various limits are strong enough to impose $W_\Delta = 0$. The limits that W_Δ should obey are

$$W_\Delta(\Lambda, \tilde{m} \rightarrow \infty) \rightarrow 0, \quad W_\Delta(\Lambda \rightarrow 0, \tilde{m}) \rightarrow 0. \quad (2.9)$$

⁷A list of warnings as to limitations of the procedure appears in [14].

When $\Lambda \rightarrow 0$, the theory becomes classical and the superpotential collapses to $W_{tree,d}$ (in some cases there is also a classical constraint). When $\tilde{m} \rightarrow \infty$, the additional degrees of freedom U become much heavier than the scale Λ of the up theory and hence are expected to decouple from the down theory, except for their influence on the renormalization of the coupling. This will make W_d depend on the down scale Λ_d . If W_Δ is indeed zero, we can integrate in the superfield U and derive the nonperturbative superpotential of the up theory.

We should remark that the W_u , derived by integrating in, is the superpotential on particular branches in the moduli space of $N = 1$ vacua – those branches which contain the heavy U region. Moreover, W_u is expected to be singular at points in the moduli space where extra degrees of freedom – not included in the procedure – become massless. We shall return to these points later.

3 Integrating in $2N_f$ doublets to pure $N = 1$, $SU(2)$ gauge theory

The nonperturbative superpotential of $N = 1$ supersymmetric $SU(2)$ gauge theory with N_f flavors ($2N_f$ doublets) can be constructed just by the use of holomorphy and symmetries [12, 13]. Yet, following refs. [3, 4], we shall also derive W_{N_f} by integrating in N_f flavors to the pure supersymmetric $SU(2)$ gauge theory.

The classical low-energy effective superpotential of the down theory vanishes, and the nonperturbative effective superpotential (due to gluino condensation) is

$$W_d(\text{pure } N = 1, SU(2)) = \pm 2\Lambda_d^3, \quad (3.1)$$

where Λ_d is the dynamically generated scale of the down theory. We now want to integrate in $2N_f$ supermultiplets in the fundamental representation, Q_i^a , $i = 1, \dots, 2N_f$, and a is a fundamental representation index. One-loop

asymptotic freedom or conformal invariance implies

$$b_1 = 6 - N_f \geq 0, \quad (3.2)$$

where $-b_1$ is the one-loop coefficient of the gauge coupling beta-function. We consider these models in the presence of masses m . The relevant gauge singlets, $X_{ij} = -X_{ji}$, are quadratic in the $N = 1$ superfield doublets, Q^a :

$$X_{ij} = \epsilon_{ab} Q_i^a Q_j^b, \quad a, b = 1, 2, \quad i, j = 1, \dots, 2N_f, \quad (3.3)$$

and, therefore,

$$W_{tree} = \frac{1}{2} \text{Tr}_{2N_f} mX \Rightarrow W_{tree,d} \equiv \frac{1}{2} \text{Tr}_{2N_f} mX|_{\langle Q \rangle} = 0. \quad (3.4)$$

By using the global symmetries, $SU(2N_f) \times U(1)_Q \times U(1)_R$, one finds that $W_\Delta = 0$ and, therefore,

$$W_u(X) = [W_d - W_{tree}]_{\langle m \rangle} = [\pm 2(\text{Pf}m)^{\frac{1}{2}} \Lambda^{\frac{6-N_f}{2}} - \frac{1}{2} \text{Tr}_{2N_f} mX]_{\langle m \rangle}. \quad (3.5)$$

Here we used the coupling constant matching (consistent with global symmetries)

$$\Lambda_d^{b_{1,d}} = (\text{Pf}m) \Lambda^{b_{1,u}}, \quad (3.6)$$

where $b_{1,d} = 6$ is minus the one-loop coefficient of the gauge coupling beta-function of the down theory, and $b_{1,u} = 6 - N_f$ is minus the one-loop coefficient of the gauge coupling beta-function of the up theory. This matching implies $W_d(m, \Lambda) = \pm 2(\text{Pf}m)^{\frac{1}{2}} \Lambda^{\frac{6-N_f}{2}}$, which we inserted in (3.5). Finally, one finds

$$W_{N_f}(X) = (2 - N_f) \Lambda^{\frac{6-N_f}{2-N_f}} (\text{Pf}X)^{\frac{1}{N_f-2}} + \frac{1}{2} \text{Tr}_{2N_f} mX, \quad (3.7)$$

where an additional tree-level superpotential has been added to W_u . For $N_f = 1$, the massless superpotential reads: $W = \Lambda^5/X$. For $N_f = 2$ ($b_1 = 4$ in eq. (3.2)), $W = 0$, and by the integrating in procedure we also get the constraint: $\text{Pf}X = \Lambda^4$. For $N_f > 2$, $W(m = 0)$ is proportional to some positive power of the classical constraint: $\text{Pf}X = 0$. Small values of Λ imply

a semi-classical limit for which the classical constraint is imposed; however, quantum corrections remove the constraint. At the $\langle X \rangle = 0$ vacuum one expects to find extra massless interacting scalars (by 't Hooft matching conditions [13] and, for $N_f > 3$, by electric-magnetic duality [10]) and, therefore, we make use of the $N_f > 2$ superpotential only in the presence of a mass matrix m ($\det m \neq 0$), which fixes the vacua at $\langle X \rangle \neq 0$.

4 Integrating in N_A triplets to $N = 1$, $SU(2)$ gauge theory with $2N_f$ doublets

We now want to derive the nonperturbative superpotential, W_{N_f, N_A} , of $N = 1$ supersymmetric $SU(2)$ gauge theory with N_A triplets and N_f flavors, by integrating in N_A triplets, Φ_α^{ab} , $\alpha = 1, \dots, N_A$, to the supersymmetric gauge theory with $2N_f$ doublets (3.7). Here a, b are fundamental representation indices, and $\Phi^{ab} = \Phi^{ba}$. As before, we treat the cases with one-loop asymptotic freedom or conformal invariance, for which

$$b_1 = 6 - N_f - 2N_A \geq 0, \quad (4.1)$$

where $-b_1$ is the one-loop coefficient of the gauge coupling beta-function.

The relevant gauge singlets we should add to X_{ij} in eq. (3.3) are $M_{\alpha\beta} = M_{\beta\alpha}$ and $Z_{ij\alpha} = Z_{ji\alpha}$:

$$\begin{aligned} M_{\alpha\beta} &= \epsilon_{aa'}\epsilon_{bb'}\Phi_\alpha^{ab}\Phi_\beta^{a'b'}, & a, b = 1, 2, & \quad \alpha, \beta = 1, \dots, N_A, \\ Z_{ij\alpha} &= \epsilon_{aa'}\epsilon_{bb'}Q_i^a\Phi_\alpha^{a'b'}Q_j^b & i, j = 1, \dots, 2N_f. \end{aligned} \quad (4.2)$$

After some algebra one can show that

$$\begin{aligned} W_{tree,d} &\equiv W_{tree}|_{\langle \Phi_\alpha \rangle} = [\text{Tr}_{N_A} \tilde{m} M + \frac{1}{\sqrt{2}} \text{Tr}_{2N_f} \lambda^\alpha Z_\alpha]|_{\langle \Phi_\alpha \rangle} \\ &= \frac{1}{8} \text{Tr}_{2N_f} (\tilde{m}^{-1})_{\alpha\beta} \lambda^\alpha X \lambda^\beta X. \end{aligned} \quad (4.3)$$

Moreover, by using the global symmetries and various limits one can show that $W_\Delta = 0$; this is done in the Appendix. Therefore,

$$W_u(M, X, Z) = [W_d + W_{tree,d} - W_{tree}]_{\langle \tilde{m} \rangle, \langle \lambda \rangle}$$

$$\begin{aligned}
&= \left[\frac{2 - N_f}{\Lambda^{\frac{6 - N_f - 2N_A}{N_f - 2}}} (\det \tilde{m})^{\frac{-2}{N_f - 2}} (\text{Pf} X)^{\frac{1}{N_f - 2}} \right. \\
&+ \frac{1}{8} \text{Tr}_{2N_f} (\tilde{m}^{-1})_{\alpha\beta} \lambda^\alpha X \lambda^\beta X \\
&\left. - \text{Tr}_{N_A} \tilde{m} M - \frac{1}{\sqrt{2}} \text{Tr}_{2N_f} \lambda^\alpha Z_\alpha \right]_{\langle \tilde{m} \rangle, \langle \lambda \rangle} \quad (4.4)
\end{aligned}$$

Here we used $W_d \equiv W_{N_f}$ of eq. (3.7), where we inserted the matching (consistent with global symmetries):

$$\Lambda_d^{b_{1,d}} = [\det(\tilde{m}/2)]^2 \Lambda^{b_{1,u}}. \quad (4.5)$$

In (4.5) $b_{1,d} = 6 - N_f$ is minus the one-loop coefficient of the gauge coupling beta-function of the down theory, and $b_{1,u} = 6 - N_f - 2N_A$ is minus the one-loop coefficient of the gauge coupling beta-function of the up theory. Finally, we obtain the superpotential⁸

$$\begin{aligned}
W_{N_f, N_A}(M, X, Z) &= (b_1 - 4) \left\{ \Lambda^{-b_1} \text{Pf} X \left[\det_{N_A}(\Gamma_{\alpha\beta}) \right]^2 \right\}^{1/(4-b_1)} \\
&+ \text{Tr}_{N_A} \tilde{m} M + \frac{1}{2} \text{Tr}_{2N_f} m X + \frac{1}{\sqrt{2}} \text{Tr}_{2N_f} \lambda^\alpha Z_\alpha, \quad (4.6)
\end{aligned}$$

where

$$\Gamma_{\alpha\beta}(M, X, Z) = M_{\alpha\beta} + \text{Tr}_{2N_f}(Z_\alpha X^{-1} Z_\beta X^{-1}). \quad (4.7)$$

Recall that in eq. (4.6), Λ is the dynamically generated scale, b_1 is given in eq. (4.1), and $\tilde{m}_{\alpha\beta}$, m_{ij} and λ_{ij}^α are the bare masses and Yukawa couplings, respectively ($\tilde{m}_{\alpha\beta} = \tilde{m}_{\beta\alpha}$, $m_{ij} = -m_{ji}$, $\lambda_{ij}^\alpha = \lambda_{ji}^\alpha$).

From eqs. (3.3), (4.2) it is clear that the determinant in W_{N_f, N_A} vanishes classically. Quantum mechanically, the constraint is removed; by taking the $\Lambda \rightarrow 0$ limit in eq. (4.6), one recovers the classical constraint $\det_{N_3}(\Gamma_{\alpha\beta}) = 0$ (if $b_1 < 4$, $N_f \neq 0$).

⁸ When $b_1 = 4$, the nonperturbative superpotential vanishes and one also obtains constraints. In the conformal case, when $b_1 = 0$, “ Λ^{-b_1} ” in (4.6) should be replaced by a function of $\tau_0 = \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}$ (the non-Abelian gauge coupling) and $\det \lambda$; this will be discussed in section 8.

Models without triplets ($N_A = 0$) were discussed in section 3. Models without doublets ($N_f = 0$) were studied in [7, 5, 15]. In these cases

$$W_{0,N_A}(M) = 2(1 - N_A)\Lambda^{\frac{N_A-3}{N_A-1}}(\det M)^{\frac{1}{N_A-1}} + \text{Tr}_{N_A}\tilde{m}M. \quad (4.8)$$

The massless $N_A = 1$ case is a pure $SU(2)$, $N = 2$ supersymmetric Yang-Mills theory. This model was considered in detail in ref. [7]. In this case, $W = 0$ (compatible with eq. (4.6)). As in the other $b_1 = 4$ case, discussed in section 3, by the integrating in procedure one also gets a constraint in this case: $M = \pm\Lambda^2$. This result can be understood because the starting point of the integrating in procedure is a pure $N = 1$ supersymmetric Yang-Mills theory. Therefore, it leads us to the points at the verge of confinement in the moduli space. These are the two singular points in the M moduli space of the theory; they are due to massless monopoles or dyons. Such excitations are not constructed out of the elementary degrees of freedom and, therefore, there is no trace for them in W . (This situation is different if $N_f \neq 0$; in this case, monopoles are different manifestations of the elementary degrees of freedom.)

The $N_f = 0$, $N_A = 2$ case is discussed in refs. [5, 15]. In this case, the superpotential in eq. (4.6) is the one presented in [5, 15] on the confinement and the oblique confinement branches (it so happens that for \tilde{m} such that $\det \tilde{m} = 0$ eq. (4.8) also describes the Coulomb phase). As in the $N_A = 1$ case, this is because the starting point of the integrating in procedure is a pure $N = 1$ supersymmetric Yang-Mills theory and, therefore, it leads us to the confining branches in the moduli space. The moduli space may also contain a non-Abelian Coulomb phase at the point $\langle M \rangle = 0$ [5, 14].

For $N_A = 3$ there is an additional Yukawa coupling that we did not consider in (4.2): the one which couples the three (antisymmetric) triplets. Therefore, we should also integrate in the additional gauge singlet $\Phi\Phi\Phi \equiv \det \Phi$. The superpotential in eq. (4.8) remains valid also in the presence of $W_{tree} = \lambda \det \Phi$ because $\det \Phi = (\det M)^{1/2}$. For this term, the Yukawa coupling, λ , replaces “ Λ^0 ” in eq. (4.8). This result coincides with the one derived in [15]. In the massless case, this theory flows to an $N = 4$ supersymmetric Yang-Mills fixed point.

In sections 5-10, we consider the supersymmetric $SU(2)$ models with $N_A \neq 0$ and $N_f \neq 0$. All the symmetries and quantum numbers of the various parameters, in particular, such as used in [7, 8], are already embodied in the superpotential W_{N_f, N_A} of eq. (4.6).

By construction, integrating out a triplet from W_{N_f, N_A} (4.6) gives the superpotential of the down theory: W_{N_f} of eq. (3.7). Moreover, integrating out a flavor from W_{N_f, N_A} gives the superpotential W_{N_f-1, N_A} and an intermediate superpotential, W_i , which vanishes in the infinite mass limit of the doublets integrated out: $W_i(m_{N_f-1, N_f} \rightarrow \infty) \rightarrow 0$.

We should remark that the singularities at $X = 0$, and the branch cuts in $\text{Pf}X$ and Γ , signal the appearance of extra massless degrees of freedom at these points (the branch cuts in Λ are due to non-Abelian effects). Those are expected, physically, due to some duality, similar to the electric-magnetic duality of refs. [10, 16]. The $SU(2)$, $N_A = 1$, N_f models fall into a lacuna in the analysis in ref. [16] of the dual models to $SU(N_c)$ systems with matter in the adjoint and fundamental representations. The results obtained here might shed some light on this gap.

Finally, to complete the survey of $SU(2)$ models obeying $b_1 \geq 0$, let us note that one can also have an infra-red non-trivial theory with a single matter superfield in the $I = 3/2$ representation. The $N_{3/2} = 1$, $N_f = 0$ theory was shown to have $W = 0$ [17]. Adding $N_f = 1$ matter results with $b_1 = 0$ in eq. (4.1). The two-loop beta function renders the theory infra-red free. As no Yukawa coupling is possible, this model is indeed infra-red free.

5 $SU(2)$ with $N_A = N_f = 1$ ($b_1 = 3$)

Before turning to the substance of this section, we should remark that we will use some notational and algebraic complications which are not necessary in the study of the $N = 1$, $SU(2)$ case with $N_A = N_f = 1$ (we shall return to a simpler manipulation of this case after discussing $SU(N_c)$ with one adjoint and one flavor in section 14). We do the manipulation here in a way similar to what we shall do in the more complicated cases of $SU(2)$ with one adjoint and several flavors.

A supersymmetric $SU(2)$ gauge theory with one triplet and one flavor has a superpotential (4.6):

$$W_{1,1} = -\Lambda^{-3}(\text{Pf}X)\Gamma^2 + \tilde{m}M + \frac{1}{2}\text{Tr } mX + \frac{1}{\sqrt{2}}\text{Tr } \lambda Z. \quad (5.1)$$

Here m and X are antisymmetric 2×2 matrices, and λ and Z are symmetric 2×2 matrices and

$$\Gamma = M + \text{Tr}(ZX^{-1})^2. \quad (5.2)$$

In ref. [5], both the classical and quantum moduli spaces were described. Both classically and quantum mechanically, the theory is generically in the Higgs/confinement phase. The classical singularity at $X = Z = M = 0$ is resolved quantum mechanically into three singularities. We will reobtain this result in detail.

We now want to find the vacua of the theory, namely, we should solve the equations of motion $\delta W_{1,1}/\delta M = \delta W_{1,1}/\delta X = \delta W_{1,1}/\delta Z = 0$, which read:

$$\tilde{m} = 2\Lambda^{-3}(\text{Pf}X)\Gamma, \quad (5.3)$$

$$m = R^{-1}(X^{-1} - 8\Gamma^{-1}X^{-1}(ZX^{-1})^2), \quad (5.4)$$

$$\frac{1}{\sqrt{2}}\lambda = 4R^{-1}\Gamma^{-1}X^{-1}ZX^{-1}, \quad (5.5)$$

where

$$R^{-1} = \Lambda^{-3}(\text{Pf}X)\Gamma^2. \quad (5.6)$$

Combining eqs. (5.4) and (5.5) we get

$$Xm + \sqrt{2}Z\lambda = R^{-1}I, \quad (5.7)$$

where I is the 2×2 identity matrix. Equation (5.5) gives

$$\frac{1}{\sqrt{2}}Z\lambda = \frac{1}{8}R\Gamma(X\lambda)^2, \quad (5.8)$$

and using (5.8), eq. (5.7) reads:

$$Y^2 + Y\nu = 2\Gamma I, \quad (5.9)$$

where

$$\nu = \frac{4}{\sqrt{2}}\lambda^{-1}m, \quad Y = \frac{1}{\sqrt{2}}R\Gamma X\lambda = 4ZX^{-1}, \quad \Gamma = M + \frac{1}{16}\text{Tr}Y^2. \quad (5.10)$$

(The form of eqs. (5.2), (5.4), (5.5), (5.7)-(5.10), as derived by variations from eq. (4.6), is N_f -independent for $N_A = 1$ and will appear again in sections 6,7,8; equations (5.1), (5.3), (5.6) contain, explicitly, the N_f dependence.)

Equations (5.10), (5.6) imply

$$\zeta_2 \equiv \det Y = -\frac{1}{2}\text{Tr}Y^2 = \frac{1}{2}\Lambda^6\Gamma^{-2}\det\lambda, \quad \text{Tr}Y = 0, \quad (5.11)$$

which implies that the characteristic polynomial of Y is

$$Y^2 + \zeta_2 I = 0. \quad (5.12)$$

Using eqs. (5.9) and (5.12) to eliminate Y gives

$$\zeta_2\nu^2 + 4\left(\Gamma + \frac{1}{2}\zeta_2\right)^2 I = 0. \quad (5.13)$$

The characteristic polynomial of ν is

$$\nu^2 + \alpha_2 I = 0, \quad \alpha_2 = \det \nu, \quad \text{Tr}\nu = 0, \quad (5.14)$$

and with (5.13) and (5.11) we obtain

$$\alpha_2\zeta_2 = 4\left(\Gamma + \frac{1}{2}\zeta_2\right)^2 = 4\Lambda^6\Gamma^{-2}(\text{Pf}m)^2. \quad (5.15)$$

From eqs. (5.10) and (5.11) we read

$$\Gamma = M - \frac{1}{8}\zeta_2, \quad (5.16)$$

and, therefore, (5.15) becomes

$$\Lambda^3\Gamma^{-1}\text{Pf}m = \Gamma + \frac{1}{2}\zeta_2 = 4M - 3\Gamma \Rightarrow 3\Gamma^2 - 4M\Gamma + \Lambda^3\text{Pf}m = 0. \quad (5.17)$$

Equations (5.16) and (5.11) imply

$$8(M - \Gamma) = \frac{1}{2}\Lambda^6\Gamma^{-2}\det\lambda \Rightarrow \Gamma^3 - M\Gamma^2 + \frac{1}{16}\Lambda^6\det\lambda = 0. \quad (5.18)$$

By combining eqs. (5.17) and (5.18) we find

$$x^3 - Mx^2 + \frac{1}{4}\Lambda^3(\text{Pfm})x - \frac{1}{64}\Lambda^6 \det \lambda = 0, \quad (5.19)$$

and

$$3x^2 - 2Mx + \frac{1}{4}\Lambda^3\text{Pfm} = 0, \quad (5.20)$$

where

$$x \equiv \frac{1}{2}\Gamma. \quad (5.21)$$

Equations (5.19) and (5.20) are the singularity conditions of an elliptic curve defined by

$$y^2 = x^3 + ax^2 + bx + c, \quad (5.22)$$

with

$$a = -M, \quad b = \frac{\Lambda^3}{4}\text{Pfm}, \quad c = -\frac{\alpha}{16}, \quad (5.23)$$

where

$$\alpha \equiv \frac{\Lambda^{2b_1}}{2^{2N_f}} \det \lambda = \frac{\Lambda^6}{4} \det \lambda. \quad (5.24)$$

Solving M in eq. (5.20), and eliminating M in eq. (5.19) we find

$$x^3 - bx - 2c = 0, \quad (5.25)$$

and

$$M = \frac{3}{2}x + \frac{b}{2}x^{-1}, \quad (5.26)$$

Therefore, we find that $W_{1,1}$ (5.1) has three (branches of) vacua, namely, the three solutions for $M(x)$ in terms of the three solutions of the cubic equation for x (5.25) – the singularities of the elliptic curve (5.22), (5.23) – and the solutions for X and Z , given by the other equations of motion; explicitly,

$$X = \frac{1}{\sqrt{2}}\tilde{m}Y\lambda^{-1}, \quad Z = \frac{1}{4\sqrt{2}}\tilde{m}Y^2\lambda^{-1}, \quad (5.27)$$

where Y is solved in terms of its invariants ζ_2 , given in eq. (5.11), up to an $SU(2N_f) = SU(2)$ rotation, which is determined by the by eq. (5.9).

These three vacua are the vacua of the theory in the Higgs-confinement phase. The phase transition points to the Coulomb branch are at $X = 0$. This happens iff the triplet superfield is massless, namely

$$X = 0 \Leftrightarrow \tilde{m} = 0. \quad (5.28)$$

The coefficients a, b, c of the elliptic curve and, in particular, its singularities, are independent of the value of X (namely, \tilde{m})⁹ and, therefore, we conclude that the elliptic curve (5.22), (5.23) defines the effective Abelian coupling, $\tau(M, m, \lambda, \Lambda)$, in the Coulomb branch.¹⁰

Equation (5.23) generalizes [5] the results of ref. [8] to *arbitrary* bare masses and Yukawa couplings. Indeed, in the $N = 2$ supersymmetric case (namely, when $\det \lambda = 1$), the result (5.23) coincides with the one obtained in ref. [8].

Finally, we should note that in eq. (5.21) we have identified a physical meaning of the parameter x : it is the composite field $\Gamma/2$, where Γ is defined in eq. (5.2).

6 $SU(2)$ **with** $N_A = 1$, $N_f = 2$ ($b_1 = 2$)

A supersymmetric $SU(2)$ gauge theory with one triplet and two flavors has a superpotential (4.6):

$$W_{2,1} = -2\Lambda^{-1}(\text{Pf}X)^{1/2}\Gamma + \tilde{m}M + \frac{1}{2}\text{Tr} mX + \frac{1}{\sqrt{2}}\text{Tr} \lambda Z. \quad (6.1)$$

Here m and X are antisymmetric 4×4 matrices, and λ and Z are symmetric 4×4 matrices and

$$\Gamma = M + \text{Tr}(ZX^{-1})^2. \quad (6.2)$$

As in section 5, we now want to find the vacua of the theory, namely, we should solve the equations of motion $\delta W_{2,1}/\delta M = \delta W_{2,1}/\delta X = \delta W_{2,1}/\delta Z =$

⁹ Note that $X = 0 \Rightarrow \det Z = 0$ in a way such that x is finite.

¹⁰Note that the singularities of an elliptic curve, and its behavior in asymptotic limits define it uniquely.

0, which read:

$$\tilde{m} = 2\Lambda^{-1}(\text{Pf}X)^{1/2}, \quad (6.3)$$

$$m = R^{-1}(X^{-1} - 8\Gamma^{-1}X^{-1}(ZX^{-1})^2), \quad (6.4)$$

$$\frac{1}{\sqrt{2}}\lambda = 4R^{-1}\Gamma^{-1}X^{-1}ZX^{-1}, \quad (6.5)$$

where

$$R^{-1} = \Lambda^{-1}(\text{Pf}X)^{1/2}\Gamma. \quad (6.6)$$

Combining eqs. (6.4) and (6.5) we get

$$Xm + \sqrt{2}Z\lambda = R^{-1}I, \quad (6.7)$$

where I is the 4×4 identity matrix. Equation (6.5) gives

$$\frac{1}{\sqrt{2}}Z\lambda = \frac{1}{8}R\Gamma(X\lambda)^2, \quad (6.8)$$

and using (6.8), eq. (6.7) reads:

$$Y^2 + Y\nu = 2\Gamma I, \quad (6.9)$$

where

$$\nu = \frac{4}{\sqrt{2}}\lambda^{-1}m, \quad Y = \frac{1}{\sqrt{2}}R\Gamma X\lambda = 4ZX^{-1}, \quad \Gamma = M + \frac{1}{16}\text{Tr}Y^2. \quad (6.10)$$

Equation (6.9) implies, in particular,

$$[Y, \nu] = 0, \quad (6.11)$$

and from (6.10), (6.6) we get

$$\zeta_2 \equiv -\frac{1}{2}\text{Tr}Y^2 = 8(M - \Gamma), \quad \zeta_4 \equiv \det Y = \frac{1}{4}\Lambda^4 \det \lambda, \quad \text{Tr}Y = \text{Tr}Y^3 = 0, \quad (6.12)$$

thus the characteristic polynomial of Y is

$$Y^4 + \zeta_2 Y^2 + \zeta_4 I = 0. \quad (6.13)$$

Using eqs. (6.9), (6.11) to eliminate Y^4 in (6.13), we get

$$(\nu^2 + \zeta_2)Y^2 - 4\Gamma\nu Y + (4\Gamma^2 + \zeta_4)I = 0. \quad (6.14)$$

Now, using eqs. (6.9), (6.11) and (6.14) to eliminate Y , we get

$$\zeta_4\nu^4 + (4\Gamma^2\zeta_2 + 8\Gamma\zeta_4 + \zeta_2\zeta_4)\nu^2 + (4\Gamma^2 + 2\Gamma\zeta_2 + \zeta_4)^2I = 0. \quad (6.15)$$

The characteristic polynomial of ν is

$$\nu^4 + \alpha_2\nu^2 + \alpha_4I = 0, \quad \alpha_4 = \det \nu, \quad \alpha_2 = -\frac{1}{2}\text{Tr}\nu^2, \quad \text{Tr}\nu = \text{Tr}\nu^3 = 0, \quad (6.16)$$

and with (6.15) and (6.12) we obtain

$$\alpha_2\zeta_4 = 4\Gamma^2\zeta_2 + 8\Gamma\zeta_4 + \zeta_2\zeta_4 = 8(M - \Gamma)(4\Gamma^2 + \zeta_4) + 8\Gamma\zeta_4, \quad (6.17)$$

and

$$(\alpha_4\zeta_4)^{1/2} = 4\Gamma^2 + 2\Gamma\zeta_2 + \zeta_4 = 16\Gamma(M - \Gamma) + 4\Gamma^2 + \zeta_4. \quad (6.18)$$

From eqs. (6.17), (6.18) we find

$$x^3 - Mx^2 + \frac{(\alpha_4\zeta_4)^{1/2} - \zeta_4}{16}x - \frac{1}{128}(\alpha_2 - 8M)\zeta_4 = 0, \quad (6.19)$$

and

$$3x^2 - 2Mx + \frac{(\alpha_4\zeta_4)^{1/2} - \zeta_4}{16} = 0, \quad (6.20)$$

where

$$x \equiv \frac{1}{2}\Gamma. \quad (6.21)$$

Equations (6.19) and (6.20) are the singularity conditions of an elliptic curve defined by

$$y^2 = x^3 + ax^2 + bx + c, \quad (6.22)$$

with

$$\begin{aligned} a &= -M, & b &= -\frac{\alpha}{4} + \frac{\Lambda^2}{4}\text{Pfm}, \\ c &= \frac{\alpha}{8}(2M + \text{Tr}(\mu^2)), \end{aligned} \quad (6.23)$$

where

$$\alpha \equiv \frac{\Lambda^{2b_1}}{2^{2N_f}} \det \lambda = \frac{\Lambda^4}{16} \det \lambda, \quad \mu = \lambda^{-1} m. \quad (6.24)$$

Here we used the explicit expressions for $\zeta_4, \alpha_4, \alpha_2$ in terms of Λ, λ, m (see eqs. (6.10), (6.12), (6.16)).

Solving M in eq. (6.20), and eliminating M in eq. (6.19) we find

$$-x^4 + \left(b + \frac{3}{4}\alpha\right)x^2 + \frac{\alpha}{4}\text{Tr}(\mu^2)x + \frac{\alpha b}{4} = 0, \quad (6.25)$$

and

$$M = \frac{3}{2}x + \frac{b}{2}x^{-1}, \quad (6.26)$$

Therefore, we find that $W_{2,1}$ (6.1) has four (branches of) vacua, namely, the four solutions for $M(x)$ in terms of the four solutions of the quartic equation for x (6.25) – the singularities of the elliptic curve (6.22), (6.23) – and the solutions for X and Z , given by the other equations of motion; explicitly,

$$X = \frac{1}{\sqrt{2}}\tilde{m}Y\lambda^{-1}, \quad Z = \frac{1}{4\sqrt{2}}\tilde{m}Y^2\lambda^{-1}, \quad (6.27)$$

where Y is solved in terms of its invariants ζ_2, ζ_4 , given in eq. (6.12), up to an $SU(2N_f) = SU(4)$ rotation, which is determined by eq. (6.9).

These four vacua are the vacua of the theory in the Higgs-confinement phase. The phase transition points to the Coulomb branch are at $X = 0$. This may happen if the triplet superfield is massless, namely, $\tilde{m} = 0$. The coefficients a, b, c of the elliptic curve and, in particular, its singularities, are independent of the value of X (namely, \tilde{m}) and, therefore, we conclude that the elliptic curve (6.22), (6.23) defines the effective Abelian coupling, $\tau(M, m, \lambda, \Lambda)$, in the Coulomb branch.

Equation (6.23) generalizes the result of ref. [8] to *arbitrary* bare masses and Yukawa couplings. Indeed, in the $N = 2$ supersymmetric case (namely, when $\lambda = \text{diag}(\lambda_1, \lambda_2)$, where λ_1, λ_2 are 2×2 matrices with $\det \lambda_1 = \det \lambda_2 = 1$, and $m = \text{diag}(m_1\epsilon, m_2\epsilon)$, where ϵ is the standard 2×2 constant antisymmetric matrix), the result (6.23) coincides with the one obtained in ref. [8].

Finally, as in the $N_f = 1$ case, we should note that in eq. (6.21) we have identified a physical meaning of the parameter x .

7 $SU(2)$ with $N_A = 1$, $N_f = 3$ ($b_1 = 1$)

A supersymmetric $SU(2)$ gauge theory with one triplet and three flavors has a superpotential (4.6):

$$W_{3,1} = -3\Lambda^{-1/3}(\text{Pf}X)^{1/3}\Gamma^{2/3} + \tilde{m}M + \frac{1}{2}\text{Tr} mX + \frac{1}{\sqrt{2}}\text{Tr} \lambda Z. \quad (7.1)$$

Here m and X are antisymmetric 6×6 matrices, and λ and Z are symmetric 6×6 matrices and

$$\Gamma = M + \text{Tr}(ZX^{-1})^2. \quad (7.2)$$

As in sections 5 and 6, we now want to find the vacua of the theory, namely, we should solve the equations of motion $\delta W_{3,1}/\delta M = \delta W_{3,1}/\delta X = \delta W_{3,1}/\delta Z = 0$, which read:

$$\tilde{m} = 2\Lambda^{-1/3}(\text{Pf}X)^{1/3}\Gamma^{-1/3}, \quad (7.3)$$

$$m = R^{-1}(X^{-1} - 8\Gamma^{-1}X^{-1}(ZX^{-1})^2), \quad (7.4)$$

$$\frac{1}{\sqrt{2}}\lambda = 4R^{-1}\Gamma^{-1}X^{-1}ZX^{-1}, \quad (7.5)$$

where

$$R^{-1} = \Lambda^{-1/3}(\text{Pf}X)^{1/3}\Gamma^{2/3}. \quad (7.6)$$

Combining eqs. (7.4) and (7.5) we get

$$Xm + \sqrt{2}Z\lambda = R^{-1}I, \quad (7.7)$$

where I is the 6×6 identity matrix. Equation (7.5) gives

$$\frac{1}{\sqrt{2}}Z\lambda = \frac{1}{8}R\Gamma(X\lambda)^2, \quad (7.8)$$

and using (7.8), eq. (7.7) reads:

$$Y^2 + Y\nu = 2\Gamma I, \quad (7.9)$$

where

$$\nu = \frac{4}{\sqrt{2}}\lambda^{-1}m, \quad Y = \frac{1}{\sqrt{2}}R\Gamma X\lambda = 4ZX^{-1}, \quad \Gamma = M + \frac{1}{16}\text{Tr}Y^2, \quad (7.10)$$

and (7.9) implies, in particular,

$$[Y, \nu] = 0. \quad (7.11)$$

From eqs. (7.10), (7.6) we get

$$\begin{aligned} \zeta_2 &\equiv -\frac{1}{2}\text{Tr}Y^2 = 8(M - \Gamma), & \zeta_4 &\equiv \frac{1}{2}\zeta_2^2 - \frac{1}{4}\text{Tr}Y^4, \\ \zeta_6 &\equiv \det Y = \frac{1}{8}\Gamma^2\Lambda^2 \det \lambda, & \text{Tr}Y &= \text{Tr}Y^3 = \text{Tr}Y^5 = 0, \end{aligned} \quad (7.12)$$

determining the characteristic polynomial of Y as

$$Y^6 + \zeta_2 Y^4 + \zeta_4 Y^2 + \zeta_6 I = 0. \quad (7.13)$$

Using eqs. (7.9), (7.11) to eliminate Y^4 and Y^6 in (7.13), we get

$$\begin{aligned} [\nu^4 + (\zeta_2 + 4\Gamma)\nu^2 + \zeta_4 + 4\Gamma^2]Y^2 &- [4\Gamma\nu^3 + \nu(8\Gamma^2 + 4\zeta_2\Gamma)]Y \\ &+ [\zeta_6 + 4\zeta_2\Gamma^2 + 4\Gamma^2\nu^2]I = 0. \end{aligned} \quad (7.14)$$

Now, using eqs. (7.9), (7.11) and (7.14) to eliminate Y , we get

$$\begin{aligned} \zeta_6\nu^6 &+ (4\Gamma^2\zeta_4 + 12\Gamma\zeta_6 + \zeta_2\zeta_6)\nu^4 \\ &+ (16\Gamma^4\zeta_2 + 32\Gamma^3\zeta_4 + 4\Gamma^2\zeta_2\zeta_4 + 36\Gamma^2\zeta_6 + 8\Gamma\zeta_2\zeta_6 + \zeta_4\zeta_6)\nu^2 \\ &+ (8\Gamma^3 + 4\Gamma^2\zeta_2 + 2\Gamma\zeta_4 + \zeta_6)^2 I = 0. \end{aligned} \quad (7.15)$$

The characteristic polynomial of ν is

$$\begin{aligned} \nu^6 + \alpha_2\nu^4 + \alpha_4\nu^2 + \alpha_6 I &= 0, & \alpha_2 &= -\frac{1}{2}\text{Tr}\nu^2, & \alpha_4 &= \frac{1}{2}\alpha_2^2 - \frac{1}{4}\text{Tr}\nu^4, \\ \alpha_6 &= \det \nu, & \text{Tr}\nu &= \text{Tr}\nu^3 = \text{Tr}\nu^5 = 0, \end{aligned} \quad (7.16)$$

and with (7.15) and (7.12) we obtain

$$\alpha_2\zeta_6 = 4\Gamma^2\zeta_4 + 12\Gamma\zeta_6 + \zeta_2\zeta_6 = 4\Gamma^2\zeta_4 + 4(2M + \Gamma)\zeta_6, \quad (7.17)$$

$$\begin{aligned} \alpha_4\zeta_6 &= 16\Gamma^4\zeta_2 + 32\Gamma^3\zeta_4 + 4\Gamma^2\zeta_2\zeta_4 + 36\Gamma^2\zeta_6 + 8\Gamma\zeta_2\zeta_6 + \zeta_4\zeta_6 \\ &= 8(M - \Gamma)(16\Gamma^4 + 4\Gamma^2\zeta_4 + 8\Gamma\zeta_6) + 32\Gamma^3\zeta_4 + 36\Gamma^2\zeta_6 + \zeta_4\zeta_6, \end{aligned} \quad (7.18)$$

and

$$(\alpha_6 \zeta_6)^{1/2} = 8\Gamma^3 + 4\Gamma^2 \zeta_2 + 2\Gamma \zeta_4 + \zeta_6 = 8\Gamma^3 + 32\Gamma^2(M - \Gamma) + 2\Gamma \zeta_4 + \zeta_6. \quad (7.19)$$

Eliminating ζ_4 from eqs. (7.17), (7.18), (7.19), and shifting $M \rightarrow M - \Lambda^2 \det \lambda / 256$ we find

$$x^3 + ax^2 + bx + c = 0, \quad (7.20)$$

and

$$3x^2 + 2ax + b = 0. \quad (7.21)$$

Here

$$x \equiv \frac{1}{2}\Gamma + \frac{\Lambda^2}{128} \det \lambda, \quad (7.22)$$

and

$$\begin{aligned} a &= -M - \alpha, \\ b &= 2\alpha M + \frac{\alpha}{2} \text{Tr}(\mu^2) + \frac{\Lambda}{4} \text{Pf} m, \\ c &= \frac{\alpha}{8} \left(-8M^2 - 4M \text{Tr}(\mu^2) - [\text{Tr}(\mu^2)]^2 + 2\text{Tr}(\mu^4) \right), \end{aligned} \quad (7.23)$$

where

$$\alpha \equiv \frac{\Lambda^{2b_1}}{2^{2N_f}} \det \lambda = \frac{\Lambda^2}{64} \det \lambda, \quad \mu = \lambda^{-1} m. \quad (7.24)$$

Here we used the explicit expressions for $\zeta_6, \alpha_6, \alpha_4, \alpha_2$ in terms of Λ, λ, m (see eqs. (7.10), (7.12), (7.16)). Equations (7.20) and (7.21) are the singularity conditions of an elliptic curve defined by

$$y^2 = x^3 + ax^2 + bx + c, \quad (7.25)$$

with coefficients a, b, c given in eqs. (7.23), (7.24).

Solving M in eq. (7.21), and eliminating M in eq. (7.20) one finds a degree 5 polynomial equation in x , $p_5(x) = 0$, and an equation for $M(x)$. Therefore, we find that $W_{3,1}$ (7.1) has five (branches of) vacua, namely, the five solutions for $M(x)$ in terms of the five solutions of the fifth order equation

for x – the singularities of the elliptic curve (7.25), (7.23) – and the solutions for X and Z , given by the other equations of motion; explicitly,

$$X = \frac{1}{\sqrt{2}}\tilde{m}Y\lambda^{-1}, \quad Z = \frac{1}{4\sqrt{2}}\tilde{m}Y^2\lambda^{-1}, \quad (7.26)$$

where Y is solved in terms of its invariants $\zeta_2, \zeta_4, \zeta_6$, given in eq. (7.12) (to find ζ_4 we use eq. (7.17)), up to an $SU(2N_f) = SU(6)$ rotation, determined by eq. (7.9).

These five vacua are the vacua of the theory in the Higgs-confinement phase. The phase transition points to the Coulomb branch are at $X = 0$. This may happen if the triplet superfield is massless, namely, $\tilde{m} = 0$. The coefficients a, b, c of the elliptic curve and, in particular, its singularities, are independent of the value of X (namely, \tilde{m}) and, therefore, we conclude that the elliptic curve (7.25), (7.23) defines the effective Abelian coupling, $\tau(M, m, \lambda, \Lambda)$, in the Coulomb branch.

Equation (7.23) generalizes the result of ref. [8] to *arbitrary* bare masses and Yukawa couplings. Indeed, in the $N = 2$ supersymmetric case (namely, when $\lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1, \lambda_2, \lambda_3$ are 2×2 matrices with $\det \lambda_1 = \det \lambda_2 = \det \lambda_3 = 1$, and $m = \text{diag}(m_1\epsilon, m_2\epsilon, m_3\epsilon)$, where ϵ is the standard 2×2 constant antisymmetric matrix), the result (7.23) coincides with the one obtained in ref. [8].

Finally, as before, we should note that in eq. (7.22) we have identified a physical meaning of the parameter x . Unlike the $N_f = 1$ and $N_f = 2$ cases, for $N_f = 3$, x is identified with $\Gamma/2$ only up to a shift by $\alpha/2$, where Γ and α are given in eqs. (7.2) and (7.24), respectively.

8 $SU(2)$ with $N_A = 1, N_f = 4$ ($b_1 = 0$)

A supersymmetric $SU(2)$ gauge theory with one triplet and four flavors has a vanishing one-loop beta-function and, therefore, will possess extra structure.

It has a superpotential (4.6): ¹¹

$$W_{4,1} = -4\Lambda^{-b_1/4}(\text{Pf}X)^{1/4}\Gamma^{1/2} + \tilde{m}M + \frac{1}{2}\text{Tr } mX + \frac{1}{\sqrt{2}}\text{Tr } \lambda Z. \quad (8.1)$$

Here m and X are antisymmetric 8×8 matrices, and λ and Z are symmetric 8×8 matrices and

$$\Gamma = M + \text{Tr}(ZX^{-1})^2. \quad (8.2)$$

As in sections 5, 6 and 7, we now want to find the vacua of the theory, namely, we should solve the equations of motion $\delta W_{4,1}/\delta M = \delta W_{4,1}/\delta X = \delta W_{4,1}/\delta Z = 0$, which read:

$$\tilde{m} = 2\Lambda^{-b_1/4}(\text{Pf}X)^{1/4}\Gamma^{-1/2}, \quad (8.3)$$

$$m = R^{-1}(X^{-1} - 8\Gamma^{-1}X^{-1}(ZX^{-1})^2), \quad (8.4)$$

$$\frac{1}{\sqrt{2}}\lambda = 4R^{-1}\Gamma^{-1}X^{-1}ZX^{-1}, \quad (8.5)$$

where

$$R^{-1} = \Lambda^{-b_1/4}(\text{Pf}X)^{1/4}\Gamma^{1/2}. \quad (8.6)$$

Combining eqs. (8.4) and (8.5) we get

$$Xm + \sqrt{2}Z\lambda = R^{-1}I, \quad (8.7)$$

where I is the 8×8 identity matrix. Equation (8.5) gives

$$\frac{1}{\sqrt{2}}Z\lambda = \frac{1}{8}R\Gamma(X\lambda)^2, \quad (8.8)$$

and using (8.8), eq. (8.7) reads:

$$Y^2 + Y\nu = 2\Gamma I, \quad (8.9)$$

¹¹ $b_1 = 0$ and, as noted before, “ Λ^{-b_1} ” in (8.1) should be replaced by a function of $\tau_0 = \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}$ (the non-Abelian gauge coupling constant) and $\det \lambda$; the issue is addressed in this section.

where

$$\nu = \frac{4}{\sqrt{2}}\lambda^{-1}m, \quad Y = \frac{1}{\sqrt{2}}R\Gamma X\lambda = 4ZX^{-1}, \quad \Gamma = M + \frac{1}{16}\text{Tr}Y^2. \quad (8.10)$$

Again, eq. (8.9) implies, in particular,

$$[Y, \nu] = 0, \quad (8.11)$$

and eqs. (8.10), (8.6) imply

$$\begin{aligned} \zeta_2 &\equiv -\frac{1}{2}\text{Tr}Y^2 = 8(M - \Gamma), & \zeta_4 &\equiv \frac{1}{2}\zeta_2^2 - \frac{1}{4}\text{Tr}Y^4, \\ \zeta_6 &\equiv \zeta_2\left(\zeta_4 - \frac{1}{3}\zeta_2^2\right) - \frac{1}{6}\text{Tr}Y^6 & \zeta_8 &\equiv \det Y = \frac{1}{16}\Gamma^4\Lambda^{2b_1}\det\lambda = 16\Gamma^4\alpha(\tau_0), \\ & & \text{Tr}Y &= \text{Tr}Y^3 = \text{Tr}Y^5 = \text{Tr}Y^7 = 0, \end{aligned} \quad (8.12)$$

Here we replaced Λ^{b_1} with a function of λ and the non-Abelian gauge coupling, τ_0 , in a way consistent with the global symmetries: ¹²

$$\Lambda^{b_1} \equiv 16\alpha(\tau_0)^{1/2}(\det\lambda)^{-1/2}, \quad (8.13)$$

where $\alpha(\tau_0)$ will be determined later. Equation (8.12) implies that the characteristic polynomial of Y is

$$Y^8 + \zeta_2 Y^6 + \zeta_4 Y^4 + \zeta_6 Y^2 + \zeta_8 I = 0. \quad (8.14)$$

Using eqs. (8.9), (8.11) to eliminate Y^4 , Y^6 and Y^8 in (8.14), we get

$$\begin{aligned} &[\nu^6 + (8\Gamma + \zeta_2)\nu^4 + (16\Gamma^2 + 4\Gamma\zeta_2 + \zeta_4)\nu^2 + 4\Gamma^2\zeta_2 + \zeta_6]Y^2 \\ &- [4\Gamma\nu^5 + (24\Gamma^2 + 4\Gamma\zeta_2)\nu^3 + (32\Gamma^3 + 8\Gamma^2\zeta_2 + 4\Gamma\zeta_4)\nu]Y \\ &+ [4\Gamma^2\nu^4 + (16\Gamma^3 + 4\Gamma^2\zeta_2)\nu^2 + 16\Gamma^4 + 4\Gamma^2\zeta_4 + \zeta_8]I = 0. \end{aligned} \quad (8.15)$$

¹² $(\det\lambda)^{-1/2}$ has the correct quantum numbers needed for the matching condition, $16\alpha^{1/2}(\det\lambda)^{-1/2}\tilde{m}^2 = \Lambda_{N_f=4, N_A=0}^2$, where $\alpha(\tau_0)$ is dimensionless, and has zero $U(1)_R \times U(1)_Q \times U(1)_\Phi$ quantum numbers.

Now, using eqs. (8.9), (8.11) and (8.15) to eliminate Y , we get

$$\begin{aligned}
\zeta_8 \nu^8 &+ (4\Gamma^2 \zeta_6 + 16\Gamma \zeta_8 + \zeta_2 \zeta_8) \nu^6 \\
&+ (16\Gamma^4 \zeta_4 + 48\Gamma^3 \zeta_6 + 80\Gamma^2 \zeta_8 + 4\Gamma^2 \zeta_2 \zeta_6 + 12\Gamma \zeta_2 \zeta_8 + \zeta_4 \zeta_8) \nu^4 \\
&+ (64\Gamma^6 \zeta_2 + 128\Gamma^5 \zeta_4 + 144\Gamma^4 \zeta_6 + 16\Gamma^4 \zeta_2 \zeta_4 + 128\Gamma^3 \zeta_8 + 32\Gamma^3 \zeta_2 \zeta_6 \\
&\quad + 36\Gamma^2 \zeta_2 \zeta_8 + 4\Gamma^2 \zeta_4 \zeta_6 + 8\Gamma \zeta_4 \zeta_8 + \zeta_6 \zeta_8) \nu^2 \\
&+ (16\Gamma^4 + 8\Gamma^3 \zeta_2 + 4\Gamma^2 \zeta_4 + 2\Gamma \zeta_6 + \zeta_8)^2 I = 0.
\end{aligned} \tag{8.16}$$

The characteristic polynomial of ν is

$$\begin{aligned}
\nu^8 + \alpha_2 \nu^6 + \alpha_4 \nu^4 + \alpha_6 \nu^2 + \alpha_8 I &= 0, \quad \alpha_2 = -\frac{1}{2} \text{Tr} \nu^2, \quad \alpha_4 = \frac{1}{2} \alpha_2^2 - \frac{1}{4} \text{Tr} \nu^4, \\
\alpha_6 &= \alpha_2 \left(\alpha_4 - \frac{1}{3} \alpha_2^2 \right) - \frac{1}{6} \text{Tr} \nu^6, \quad \alpha_8 = \det \nu, \\
\text{Tr} \nu &= \text{Tr} \nu^3 = \text{Tr} \nu^5 = \text{Tr} \nu^7 = 0,
\end{aligned} \tag{8.17}$$

and with (8.16) and (8.12) we obtain

$$\alpha_2 \zeta_8 = 4\Gamma^2 \zeta_6 + 16\Gamma \zeta_8 + \zeta_2 \zeta_8, \tag{8.18}$$

$$\alpha_4 \zeta_8 = 16\Gamma^4 \zeta_4 + 48\Gamma^3 \zeta_6 + 80\Gamma^2 \zeta_8 + 4\Gamma^2 \zeta_2 \zeta_6 + 12\Gamma \zeta_2 \zeta_8 + \zeta_4 \zeta_8, \tag{8.19}$$

$$\begin{aligned}
\alpha_6 \zeta_8 &= 64\Gamma^6 \zeta_2 + 128\Gamma^5 \zeta_4 + 144\Gamma^4 \zeta_6 + 16\Gamma^4 \zeta_2 \zeta_4 + 128\Gamma^3 \zeta_8 \\
&+ 32\Gamma^3 \zeta_2 \zeta_6 + 36\Gamma^2 \zeta_2 \zeta_8 + 4\Gamma^2 \zeta_4 \zeta_6 + 8\Gamma \zeta_4 \zeta_8 + \zeta_6 \zeta_8,
\end{aligned} \tag{8.20}$$

and

$$(\alpha_8 \zeta_8)^{1/2} = 16\Gamma^4 + 8\Gamma^3 \zeta_2 + 4\Gamma^2 \zeta_4 + 2\Gamma \zeta_6 + \zeta_8. \tag{8.21}$$

Inserting in (8.18)-(8.21) the explicit expressions of ζ_8 and α_8 in eqs. (8.12) and (8.17), respectively, then eliminating ζ_6 and ζ_4 , and after some algebra we find

$$\begin{aligned}
&256(\alpha - 1)^2 \Gamma^3 + 64(\alpha - 1)[8M(\alpha + 1) - \alpha_2 \alpha] \Gamma^2 \\
&+ 16[(\alpha + 1) \Lambda^{b_1} \text{Pfm} - 4\alpha(\alpha_4 + 8M(8M - \alpha_2))] \Gamma \\
&+ 16[\alpha_2 \alpha + 8M(1 - \alpha)] \left[\frac{-\alpha}{(\alpha - 1)^2} \Lambda^{b_1} \text{Pfm} + \frac{\alpha(\alpha + 1)}{(\alpha - 1)^2} [\alpha_4 + 8M(8M - \alpha_2)] \right] \\
&+ \frac{8\alpha_2 \alpha}{(\alpha - 1)^2} [(\alpha + 1) \Lambda^{b_1} \text{Pfm} - 4\alpha(\alpha_4 + 8M(8M - \alpha_2))] - 16\alpha_6 \alpha = 0,
\end{aligned} \tag{8.22}$$

and

$$\begin{aligned} & 768(\alpha - 1)^2\Gamma^2 + 128(\alpha - 1)[8M(\alpha + 1) - \alpha_2\alpha]\Gamma \\ & + 16(\alpha + 1)\Lambda^{b_1}\text{Pfm} - 64\alpha[\alpha_4 + 8M(8M - \alpha_2)] = 0. \end{aligned} \quad (8.23)$$

Shifting and rescaling M : $M \rightarrow \beta^2\left[M - \frac{\alpha}{\beta^2(\alpha-1)}\text{Tr}\mu^2\right]$, where $\beta \equiv \beta(\tau_0)$ is to be determined, we find that (8.22) and (8.23) become

$$x^3 + ax^2 + bx + c = 0, \quad (8.24)$$

and

$$3x^2 + 2ax + b = 0. \quad (8.25)$$

Here

$$x \equiv \frac{1}{\beta^4}\left[\Gamma - \frac{4\alpha}{(\alpha - 1)^2}\text{Tr}\mu^2\right], \quad \mu = \lambda^{-1}m, \quad (8.26)$$

and

$$\begin{aligned} a &= \frac{1}{\beta^2}\left\{2\frac{\alpha+1}{\alpha-1}M + \frac{8}{\beta^2}\frac{\alpha}{(\alpha-1)^2}\text{Tr}(\mu^2)\right\}, \\ b &= \frac{1}{\beta^4}\left\{-16\frac{\alpha}{(\alpha-1)^2}M^2 + \frac{32}{\beta^2}\frac{\alpha(\alpha+1)}{(\alpha-1)^3}M\text{Tr}(\mu^2) \right. \\ &\quad \left. - \frac{8}{\beta^4}\frac{\alpha}{(\alpha-1)^2}\left[(\text{Tr}(\mu^2))^2 - 2\text{Tr}(\mu^4)\right] + \frac{4}{\beta^4}\frac{(\alpha+1)\Lambda^{b_1}}{(\alpha-1)^2}\text{Pfm}\right\}, \\ c &= \frac{1}{\beta^6}\left\{-32\frac{\alpha(\alpha+1)}{(\alpha-1)^3}M^3 + \frac{32}{\beta^2}\frac{\alpha(\alpha+1)^2}{(\alpha-1)^4}M^2\text{Tr}(\mu^2) \right. \\ &\quad \left. + M\left[-\frac{16}{\beta^4}\frac{\alpha(\alpha+1)}{(\alpha-1)^3}\left((\text{Tr}(\mu^2))^2 - 2\text{Tr}(\mu^4)\right) + \frac{32}{\beta^4}\frac{\alpha\Lambda^{b_1}}{(\alpha-1)^3}\text{Pfm}\right] \right. \\ &\quad \left. - \frac{32}{\beta^6}\frac{\alpha}{(\alpha-1)^2}\left[\text{Tr}(\mu^2)\text{Tr}(\mu^4) - \frac{1}{6}(\text{Tr}(\mu^2))^3 - \frac{4}{3}\text{Tr}(\mu^6)\right]\right\}. \end{aligned} \quad (8.27)$$

Equations (8.24) and (8.25) are the singularity conditions of an elliptic curve defined by

$$y^2 = x^3 + ax^2 + bx + c, \quad (8.28)$$

with coefficients a, b, c given in eqs. (8.27). The singularity condition for the elliptic curve is equivalent to the vanishing condition of the discriminant:

$$\Delta = 4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2 = 0, \quad (8.29)$$

and, moreover, from eqs. (8.24), (8.25) we can get

$$x = \frac{ab - 9c}{6b - 2a^2}. \quad (8.30)$$

$\Delta(M)$ is a polynomial in M of degree 6 and, therefore, there are six (branches of) vacua ($x \equiv x(M)$ is given by eq. (8.30), and the solutions for X and Z are given by the other equations of motion, as was done in sections 5,6,7 for $N_f < 4$). These are the vacua of the theory in the Higgs/confinement phase. The phase transition points to the Coulomb branch are at $X = 0$. This may happen if the triplet superfield is massless, namely, $\tilde{m} = 0$. The coefficients a, b, c of the elliptic curve and, in particular, its singularities, are independent of the value of X (namely, \tilde{m}) and, therefore, we conclude that the elliptic curve (8.28), (8.27) defines the effective Abelian coupling, $\tau(M, m, \lambda, \Lambda)$, in the Coulomb branch.

We should now determine α and β . They are functions of τ_0 , the non-Abelian gauge coupling constant; comparison with ref. [8] gives ¹³

$$\alpha(\tau_0) \equiv \frac{\Lambda^{2b_1}}{2^{2N_f}} \det \lambda = \left(\frac{\theta_2^2 - \theta_3^2}{\theta_2^2 + \theta_3^2} \right)^2, \quad \beta(\tau_0) = \frac{\sqrt{2}}{\theta_2 \theta_3}, \quad (8.31)$$

where

$$\theta_2(\tau_0) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i \tau_0 n^2}, \quad \theta_3(\tau_0) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau_0 n^2}, \quad \tau_0 = \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}. \quad (8.32)$$

Equation (8.27) generalizes the result of ref. [8] to *arbitrary* bare masses and Yukawa couplings. As in the other cases, all the symmetries and quantum numbers of the various parameters, as used in [7, 8], are already embodied in the superpotential $W_{4,1}$ of eq. (8.1).

The S -duality symmetry is valid in the $N_A = 1, N_f = 4$ theories for *arbitrary* λ, m , similar to the $SL(2, \mathbb{Z})$ invariance in the presence of masses

¹³ To compare eq. (8.27) with the $N = 2$ supersymmetric case in ref. [8] we need to take $m = \text{diag}(m_1\epsilon, m_2\epsilon, m_3\epsilon, m_4\epsilon)$ and $\lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_I , $I = 1, 2, 3, 4$, are 2×2 matrices with $\det(\lambda_I) = 1$. In this case, $\text{Tr}(\mu^2) = -2 \sum_{I=1}^4 m_I^2$, $(\text{Tr}(\mu^2))^2 - 2\text{Tr}(\mu^4) = 8 \sum_{I < J} m_I^2 m_J^2$, $\text{Tr}(\mu^2)\text{Tr}(\mu^4) - \frac{1}{6}(\text{Tr}(\mu^2))^3 - \frac{4}{3}\text{Tr}(\mu^6) = 8 \sum_{I < J < K} m_I^2 m_J^2 m_K^2$.

discussed in ref. [8]. The $SL(2, Z)$ transformations map τ_0 to $(a\tau_0 + b)(c\tau_0 + d)^{-1}$, $a, b, c, d \in Z$, $ad - bc = 1$ [18]. Combined with triality (which acts on μ), it leaves the elliptic curve invariant.

Finally, we should note again that in eq. (8.26) we have identified a physical meaning of the parameter x . Unlike the $N_f < 4$ cases, for $N_f = 4$, x is identified with Γ only up to a shift by a μ -dependent and a τ_0 -dependent function (which vanish at $m = 0$), and a rescaling by a τ_0 -dependent function.

9 $SU(2)$ with $N_A = 2$, $N_f = 1$ ($b_1 = 1$)

The $N = 1$ supersymmetric $SU(2)$ gauge theory with two triplets ($N_A = 2$, $N_f = 0$) was discussed in section 4. In this section, we consider a supersymmetric $SU(2)$ gauge theory with two triplets and one flavor. The superpotential (4.6) is

$$W_{1,2} = -3\Lambda^{-1/3}(\text{Pf}X)^{1/3}(\det \Gamma)^{2/3} + \text{Tr}_{N_A} \tilde{m}M + \frac{1}{2}\text{Tr}_{2N_f} mX + \frac{1}{\sqrt{2}}\text{Tr}_{2N_f} \lambda^\alpha Z_\alpha. \quad (9.1)$$

Here m and X are antisymmetric 2×2 matrices, λ^α and Z_α are symmetric 2×2 matrices, $\alpha = 1, 2$, \tilde{m} , M are 2×2 symmetric matrices and

$$\Gamma_{\alpha\beta} = M_{\alpha\beta} + \text{Tr}(Z_\alpha X^{-1} Z_\beta X^{-1}). \quad (9.2)$$

We now want to find the vacua of the theory, namely, we should solve the equations of motion $\delta W_{1,2}/\delta M_{\alpha\beta} = \delta W_{1,2}/\delta X = \delta W_{1,2}/\delta Z_\alpha = 0$. The procedure is similar to the $N_A = N_f = 1$ case, with the additional complication induced by the matrix structure of Γ ; the equations of motion read:

$$\tilde{m}_{\alpha\beta} = 2R^{-1}(\Gamma^{-1})^{\alpha\beta}, \quad (9.3)$$

$$m = R^{-1}(X^{-1} - 8(\Gamma^{-1})^{\alpha\beta} X^{-1} Z_\alpha X^{-1} Z_\beta X^{-1}), \quad (9.4)$$

$$\frac{1}{\sqrt{2}}\lambda^\alpha = 4R^{-1}(\Gamma^{-1})^{\alpha\beta} X^{-1} Z_\beta X^{-1}, \quad (9.5)$$

where

$$R^{-1} = \Lambda^{-1/3}(\text{Pf}X)^{1/3}(\det \Gamma)^{2/3}. \quad (9.6)$$

Combining eqs. (9.4) and (9.5) we get

$$Xm + \sqrt{2}Z_\alpha\lambda^\alpha = R^{-1}I, \quad (9.7)$$

where I is the 2×2 identity matrix.

Inserting (9.5) in (9.2), and using $X = \epsilon \text{Pf} X$, where ϵ is the standard antisymmetric 2×2 matrix, we obtain

$$M = \Gamma - \left(\frac{R \text{Pf} X}{4}\right)^2 \Gamma S \Gamma, \quad (9.8)$$

where

$$S^{\alpha\beta} = \frac{1}{2} \text{Tr}(\epsilon \lambda^\alpha \epsilon \lambda^\beta). \quad (9.9)$$

Using eq. (9.3) we find

$$M_{\alpha\beta} = \frac{1}{2} R (\det \Gamma) \mu_{\alpha\beta} - \left(\frac{R^2 \text{Pf} X \det \Gamma}{8}\right)^2 \hat{S}_{\alpha\beta}, \quad (9.10)$$

where

$$\mu_{11} = \tilde{m}_{22}, \quad \mu_{22} = \tilde{m}_{11}, \quad \mu_{12} = \mu_{21} = -\tilde{m}_{12}, \quad (9.11)$$

and

$$\begin{aligned} \hat{S}_{11} &= S^{11} \tilde{m}_{22}^2 + S^{22} \tilde{m}_{12}^2 - 2S^{12} \tilde{m}_{12} \tilde{m}_{22}, \\ \hat{S}_{12} &= -S^{11} \tilde{m}_{12} \tilde{m}_{22} - S^{22} \tilde{m}_{12} \tilde{m}_{11} + S^{12} (\tilde{m}_{11} \tilde{m}_{22} + \tilde{m}_{12}^2), \\ \hat{S}_{22} &= S^{11} \tilde{m}_{12}^2 + S^{22} \tilde{m}_{11}^2 - 2S^{12} \tilde{m}_{12} \tilde{m}_{11} = -\det(\tilde{m}_{12} \lambda^1 - \tilde{m}_{11} \lambda^2). \end{aligned} \quad (9.12)$$

From eqs. (9.5), (9.7), (9.9) we get

$$\frac{1}{4} R^2 (\text{Pf} X)^2 \Gamma_{\alpha\beta} S^{\alpha\beta} = 1 - R \text{Pf}(mX), \quad (9.13)$$

and using eq. (9.3) we obtain

$$\frac{1}{8} R^3 (\text{Pf} X)^2 (\det \Gamma) \hat{S}_{22} + \frac{1}{2} R (\text{Pf} X)^2 S^{11} = \tilde{m}_{11} [1 - R \text{Pf}(mX)]. \quad (9.14)$$

Using eq. (9.6), and after some algebra, eqs. (9.10) and (9.14) read:

$$\begin{aligned} M_{11} &= \frac{1}{2}R(\det \Gamma)\tilde{m}_{22} - \left(\frac{R^2\text{Pf}X \det \Gamma}{8}\right)^2 \hat{S}_{11}, \\ M_{12} &= -\frac{1}{2}R(\det \Gamma)\tilde{m}_{12} - \left(\frac{R^2\text{Pf}X \det \Gamma}{8}\right)^2 \hat{S}_{12}, \end{aligned} \quad (9.15)$$

$$\begin{aligned} &x^3 - M_{22}x^2 + \frac{1}{16}\Lambda\tilde{m}_{11}^2(\text{Pf}m)x \\ &+ \left(\frac{\Lambda\tilde{m}_{11}}{32}\right)^2 \hat{S}_{22} - \frac{1}{512}\Lambda^2\tilde{m}_{11}^2 \det \tilde{m} \det \lambda^1 = 0, \end{aligned} \quad (9.16)$$

$$3x^2 - 2M_{22}x + \frac{1}{16}\Lambda\tilde{m}_{11}^2\text{Pf}m - \left(\frac{1}{512}\Lambda^2\tilde{m}_{11}^2 \det \tilde{m} \det \lambda^1\right)\frac{1}{x} = 0, \quad (9.17)$$

where

$$x \equiv \frac{1}{4}R(\det \Gamma)\tilde{m}_{11} = \frac{\tilde{m}_{11}}{2(\det \tilde{m})^{1/2}}(\det \Gamma)^{1/2}. \quad (9.18)$$

The solutions of eqs. (9.15), (9.16) and (9.17) determine the vacua of the theory; there are three (branches of) vacua. (Integrating out the doublets, namely, taking $m \rightarrow \infty$ keeping $\Lambda\text{Pf}m$ fixed, one is left with two vacua). We now want to consider the phase transition points from the confinement branch to the Coulomb branch. This happens at vacua where $\langle \det M \rangle = 0$, namely, when $\det \tilde{m} = 0$. Explicitly, without loss of generality, we study the case when the triplet Φ_2 is massless:

$$\tilde{m}_{22} = \tilde{m}_{12} = 0, \quad (9.19)$$

so inserting (9.19) in eq. (9.15) implies

$$M_{11} = M_{12} = 0 \Rightarrow \det M = 0. \quad (9.20)$$

Inserting (9.19) in eqs. (9.16) and (9.17) turn them into

$$x^3 + ax^2 + bx + c = 0, \quad (9.21)$$

and

$$3x^2 + 2ax + b = 0, \quad (9.22)$$

respectively, with

$$a = -M_{22}, \quad b = \frac{\Lambda \tilde{m}_{11}^2}{16} \text{Pf} m, \quad c = -\left(\frac{\Lambda \tilde{m}_{11}^2}{32}\right)^2 \det \lambda^2. \quad (9.23)$$

These are now the singularity equations of an elliptic curve

$$y^2 = x^3 + ax^2 + bx + c. \quad (9.24)$$

This curve defines the effective Abelian coupling in the Coulomb branch.

We should note that when $m = 0$ and $\tilde{m}_{11} \rightarrow 0$ (or $\det \lambda^2 \rightarrow 0$), the three singularities of the elliptic curve degenerate; when $m \neq 0$ and $\tilde{m}_{11} \rightarrow 0$, two out of the three singularities degenerate. This leads to a vacuum where mutually non-local degrees of freedom are massless, similar to the situation in pure $N = 2$ supersymmetric $SU(3)$ gauge theories, considered in [9]. Such a point might be interpreted as a non-Abelian Coulomb phase [10]. (Integrating out the doublets, namely, taking $m \rightarrow \infty$ keeping $\Lambda \text{Pf} m$ fixed, a similar phenomenon happens for the two vacua of the $N_A = 2$, $N_f = 0$ theory: they collapse into a single vacuum where a monopole and a dyon are mutually massless [14]).

The reduction from $N_A = 2$ to $N_A = 1$ is obtained by the matching

$$\Lambda_d^3 = \frac{1}{4} \Lambda \tilde{m}_{11}^2. \quad (9.25)$$

Equation (9.23) becomes:

$$a = -M_{22}, \quad b = \frac{\Lambda_d^3}{4} \text{Pf} m, \quad c = -\frac{\Lambda_d^6}{64} \det \lambda^2. \quad (9.26)$$

This result is exactly the one obtained in the $N_A = N_f = 1$ case in eq. (5.23) (with M_{22} replacing M , λ^2 replacing λ and Λ_d replacing Λ).

Finally, we should remark that, unlike the $N_A = 1$ cases, eqs. (9.16), (9.17) are not the singularity conditions of an elliptic curve, in general. However, as mentioned before, they do become the singularity conditions of an elliptic curve when $\det \tilde{m} = 0$ (as expected, physically, in a Coulomb phase),

or when $\det \lambda^1 = 0$ or when $\det \lambda^2 = 0$.¹⁴ Moreover, in eq. (9.18) we have identified a physical meaning of the parameter x .

10 $SU(2)$ with $N_A = N_f = 2$ ($b_1 = 0$) and other $b_1 = 0$ theories revisited

So far, the (massless) cases with $b_1 = 0$ we have studied ($N_A = 3, N_f = 0$ in section 4, and $N_A = 1, N_f = 4$ in section 8) were interacting conformal theories in the infra-red; the Yukawa couplings would flow to values where the supersymmetry is enhanced ($N = 4$, and $N = 2$ with four flavors, respectively), and these theories have an infra-red fixed line of marginal deformations. However, it is possible that a $b_1 = 0$ theory flows to a free theory in an infra-red fixed point. In this section, we consider this issue following ref. [11] and, in particular, we argue that the $N_A = N_f = 2$ case is a free theory in the infra-red.

We start by considering an $N = 1$ supersymmetric gauge theory with a simple gauge group, G , and a gauge coupling, g , and with a tree-level superpotential

$$W_{tree} = \lambda \phi_1 \phi_2 \cdots \phi_n, \quad (10.1)$$

where the superfield ϕ_i is in the representation R_i of G . The beta-functions are [19]

$$\beta_\lambda \equiv \frac{\partial \lambda(\mu)}{\partial \ln \mu} = \lambda(\mu) \left(-3 + \sum_{k=1}^n d(\phi_k) + \frac{1}{2} \sum_{k=1}^n \gamma(\phi_k) \right) \quad (10.2)$$

$$\beta_g \equiv \frac{\partial g(\mu)}{\partial \ln \mu} = -f(g(\mu)) \left([3C(G) - \sum_i S(R_i)] + \sum_i S(R_i) \gamma(\phi_i) \right). \quad (10.3)$$

Here f is a function of g , $d(\phi_i)$ is the naive dimension of ϕ_i , $\gamma(\phi_i)$ is the anomalous dimension of ϕ_i , $C(G)$ is the quadratic Casimir of the adjoint

¹⁴ Equations (9.16), (9.17) can be reorganized in such a way that they become, manifestly, the singularity conditions of an elliptic curve also when $\det \lambda^2 = 0$; equivalently, one can use the $1 \leftrightarrow 2$ symmetry to interchange 1 and 2 indices.

representation of G : $f_{cd}^a f^{bcd} \equiv C(G)\delta^{ab}$, and $S(R_i)$ is the Dynkin index of the representation R_i : $\text{Tr}_R(T^a T^b) \equiv S(R)\delta^{ab}$; $3C(G) - \sum_i S(R_i) = b_1$.

Consider $G = SU(2)$ with $N_A \neq 0$ triplets and N_f doublets such that $b_1 = 0$, namely, $(N_f, N_A) = (0, 3), (2, 2)$ or $(4, 1)$, and with a superpotential of the schematic form:

$$W_{tree} = \lambda \Phi Q Q, \quad (10.4)$$

where Φ is a triplet and Q is a doublet (triplet) if $N_A \neq 3$ ($N_A = 3$). (The superpotential includes several terms of this form if $N_A \neq 3$). This is the massless case and, moreover, the operators in W are marginal, namely, $\sum_{k=1}^3 d(\phi_k) = d(\Phi) + 2d(Q) = 3$. Therefore, for any Yukawa coupling of the form (10.4) the beta-function (10.2) reads:

$$\beta_\lambda = \frac{1}{2} \lambda [\gamma(\Phi) + 2\gamma(Q)]. \quad (10.5)$$

The gauge-coupling beta-function depends on the numbers N_A, N_f ; we consider the $b_1 = 0$ cases and, therefore, eq. (10.3) reads:

$$\beta_g = -f(g)[2N_A\gamma(\Phi) + N_f\gamma(Q)]. \quad (10.6)$$

We now consider case by case:

- $N_A = 3, N_f = 0$: Q and Φ are triplets and, therefore, $\gamma(Q) = \gamma(\Phi)$, which implies: $\beta_g \sim \beta_\lambda \sim \gamma(\Phi)$. Thus we get one equation in two variables, so it has a fixed line of solutions: the space of $N = 4, SU(2)$ theories (with different gauge couplings).
- $N_A = 1, N_f = 4$: $\beta_g \sim \beta_\lambda \sim \gamma(\Phi) + 2\gamma(Q)$. Therefore, we get one equation in two variables, so it has a fixed line of solutions: the space of $N = 2, SU(2)$ theories with four flavors (with different gauge couplings).
- $N_A = N_f = 2$: $\beta_g \sim 2\gamma(\Phi) + \gamma(Q)$, $\beta_\lambda \sim \gamma(\Phi) + 2\gamma(Q)$. Therefore, we get two equations in two variables, and we expect a discrete set of fixed points.

In all cases, $g = \lambda = 0$ solves the equations and, for the $N_A = N_f = 2$ theory, no other solutions exists in a small enough neighborhood of this point. The sign of β_g is such that it flows towards $g \rightarrow 0$ in the infra-red. Therefore, we argue that the $N_A = N_f = 2$, $SU(2)$ theory is infra-red free,¹⁵ and there is not much more to say about it.

11 $SU(N_c)$ with $N_A = 1$, $N_f = 0$ ($b_1 = 2N_c$)

In the following sections we discuss $SU(N_c)$ ($N_c > 2$) with N_A matter supermultiplets in the adjoint representation, and N_f flavors (N_f fundamentals and N_f anti-fundamentals). We begin in this section by integrating in a single adjoint matter ($N_A = 1$) to pure $N = 1$ supersymmetric $SU(N_c)$ gauge theory.

The down theory has a nonperturbative superpotential (due to gluino condensation):

$$W_d(\text{pure } N = 1, SU(N_c)) = N_c(\Lambda_d^{b_{1,d}})^{1/N_c}, \quad (11.1)$$

where $b_{1,d} = 3N_c$ is minus the one-loop coefficient of the gauge coupling beta-function of the down theory. We now want to integrate in a single supermultiplet in the adjoint representation, Φ^{ab} , $a, b = 1, \dots, N_c$, $\text{Tr}\Phi = 0$. The relevant gauge singlets, U_k , are the $N_c - 1$ Casimirs of $SU(N_c)$:

$$U_k = \text{Tr}\Phi^k, \quad k = 2, \dots, N_c, \quad (11.2)$$

and, therefore,

$$W_{tree} = \sum_{k=2}^{N_c} m_k U_k \Rightarrow W_{tree,d} \equiv \sum_{k=2}^{N_c} m_k \text{Tr}\Phi^k|_{\langle\Phi\rangle}. \quad (11.3)$$

Extremizing W_{tree} with respect to Φ , one should recall that Φ is traceless and, therefore, $\partial U_k / \partial \Phi^{ab} = k(\Phi^{k-1})^{ba} - (k/N_c)U_{k-1}\delta_{ab}$. From (11.3) we see that $W_{tree,d}$ is not unique, but a set of solutions to polynomial equations

¹⁵ A related fact is that (unlike the $N_A = 1$, $N_f = 4$ case) it is impossible to construct the matching “ Λ^{b_1} ” = $\alpha(\tau_0)f(\lambda^\alpha)$ in a way that respects the global symmetries.

in U_k , corresponding to different classical vacua and, therefore, there could be several branches. We will argue that a physical branch is found when $W_{tree,d} = 0$.¹⁶

Let us define the rescaled fields, φ , with $(0,0)$ $U(1)_\Phi \times U(1)_R$ quantum numbers

$$\varphi \equiv \frac{m_{N_c}}{m_{N_c-1}} \Phi, \quad (11.4)$$

and the $N_c - 2$ parameters, t_k , with $(0,0)$ $U(1)_\Phi \times U(1)_R$ quantum numbers

$$t_1 = \Lambda \frac{m_{N_c}}{m_{N_c-1}}, \quad t_k = m_k \frac{m_{N_c}^{N_c-1-k}}{m_{N_c-1}^{N_c-k}}, \quad k = 2, \dots, N_c - 2, \quad (11.5)$$

where Λ is the dynamically generated scale of the up theory. We find that the $N_c - 3$ parameters t_2, \dots, t_{N_c-2} are involved in the minimization of W_{tree} :

$$W_{tree,d} = \tau \left[\sum_{k=2}^{N_c-2} t_k \text{Tr} \varphi^k + \text{Tr} \varphi^{N_c-1} + \text{Tr} \varphi^{N_c} \right]_{\langle \varphi \rangle} = \tau f_{tree,d}(t_2, \dots, t_{N_c-2}), \quad (11.6)$$

where the parameter τ has $(0,2)$ $U(1)_\Phi \times U(1)_R$ quantum numbers

$$\tau \equiv \frac{m_{N_c}^{N_c}}{m_{N_c-1}^{N_c-1}}. \quad (11.7)$$

The up theory has a nonperturbative superpotential

$$\begin{aligned} W_u(U_k) &= [W_d + W_{tree,d} + W_\Delta - W_{tree}]_{\langle m_k \rangle} \\ &= \left[(\Lambda^{b_1})^{1/N_c} m_2 + \tau f(t) - \sum_{k=2}^{N_c} m_k U_k \right]_{\langle m_k \rangle}, \end{aligned} \quad (11.8)$$

where $b_1 = 2N_c$,¹⁷ and

$$t \equiv (t_1, \dots, t_{N_c-2}), \quad f(t) \equiv f_{tree,d}(t_2, \dots, t_{N_c-2}) + f_\Delta(t). \quad (11.9)$$

¹⁶ It is possible that the other solutions also lead to physical branches – associated with other classical vacua, and maybe with vacua such as those discussed in [9] – whose $W_{tree,d}$ and W_Δ vanish when $m_k \rightarrow \infty$ for $k \neq 2$; it is plausible that such branches involve these m_k , with $k > 2$, also in the matching conditions, in addition to m_2 [20].

¹⁷ In eq. (11.8) we wrote $(\Lambda^{2N_c})^{1/N_c}$ instead of Λ^2 , to keep the N_c possibilities corresponding to the N_c -roots of the identity: $(\Lambda^{2N_c})^{1/N_c} = \theta_{N_c}^i \Lambda^2$, $i = 0, 1, \dots, N_c - 1$, $\theta_{N_c} \equiv \exp(2\pi i/N_c)$.

In eq. (11.8) we used the $U(1)_\Phi \times U(1)_R$ global symmetries to write

$$W_\Delta = \tau f_\Delta(t), \quad (11.10)$$

and we used the matching

$$\Lambda_d^{b_{1,d}} = \left(\frac{m_2}{N_c}\right)^{N_c} \Lambda^{b_1}, \quad (11.11)$$

where recall that $b_{1,d} = 3N_c$ and $b_1 \equiv b_{1,u} = 2N_c$.

Unlike the $SU(2)$ case, when $N_c > 2$ the limits $\Lambda \rightarrow 0$ and $m_2 \rightarrow \infty$ are not enough to impose $W_\Delta = 0$. However, it is shown in the Appendix that imposing in addition the condition to have a physical branch with a discrete number of vacua implies that on such a branch $W_{tree,d} = 0$, and $W_\Delta = 0$. This implies that $W_u = 0$ with the constraint:

$$\begin{aligned} U_2 = (\Lambda^{2N_c})^{1/N_c} &\equiv \theta_{N_c}^n \Lambda^2, & n = 0, 1, \dots, N_c - 1, & \quad \theta_{N_c} = e^{\frac{2\pi i}{N_c}}, \\ U_k &= 0, & k = 3, \dots, N_c. & \end{aligned} \quad (11.12)$$

These correspond to the N_c “ $SU(N_c)$ vacua” which transform to each other under a Z_{N_c} transformation acting on the moduli space.

12 $SU(N_c)$ with $N_A = 0$, $N_c > N_f \neq 0$ ($b_1 = 3N_c - N_f$)

The nonperturbative superpotential, $W_{N_f,0}$, of $N = 1$ supersymmetric $SU(N_c)$ gauge theory with $N_c > 2$ and $N_f < N_c$ flavors (when $N_f \geq N_c$ there are also baryons in the theory), can be constructed [12, 13] just by the use of holomorphy and symmetries, or by integrating in N_f flavors to the pure $N = 1$ supersymmetric $SU(N_c)$ gauge theory with superpotential (11.1). This is done similarly to what we described for $SU(2)$ in section 3; here we only present the result.

The N_f flavors are N_f matter supermultiplets in the fundamental representation, Q_i^a , and N_f supermultiplets in the anti-fundamental, $\bar{Q}_a^{\bar{i}}$, $a =$

$1, \dots, N_c$, $i, \bar{i} = 1, \dots, N_f$. The relevant gauge singlets, $X_i^{\bar{i}}$, are given in terms of Q, \bar{Q} by

$$X_i^{\bar{i}} = Q_i^a \bar{Q}_a^{\bar{i}}. \quad (12.1)$$

The superpotential reads

$$W_{N_f,0}(X) = (N_c - N_f) \Lambda^{\frac{3N_c - N_f}{N_c - N_f}} (\det X)^{\frac{1}{N_f - N_c}} + \text{Tr}_{N_f} m X. \quad (12.2)$$

13 $SU(N_c)$ with $N_A = N_f = 1$ ($b_1 = 2N_c - 1$)

To derive the nonperturbative superpotential, $W_{1,1}$, of $N = 1$ supersymmetric $SU(N_c)$ gauge theory with one supermultiplet in the adjoint representation and one flavor, we integrate in an adjoint matter to the supersymmetric $SU(N_c)$ theory with $N_f = 1$. The down theory superpotential is given by $W_{1,0}(X)$ in eq. (12.2):

$$W_d = (b_1 - N_c) \Lambda^{\frac{b_1}{b_1 - N_c}} \left(\frac{m_2}{N_c} \right)^{\frac{N_c}{b_1 - N_c}} X^{\frac{1}{N_c - b_1}}. \quad (13.1)$$

Here we used the matching (11.11) with $b_{1,d} = 3N_c - 1$ and $b_1 \equiv b_{1,u} = 2N_c - 1$.

The relevant gauge singlets we should add to $X = Q^a \bar{Q}_a$ in the up theory are U_k , given in eq. (11.2), and Z :¹⁸

$$Z = Q^a \Phi_a^b \bar{Q}_b, \quad (13.2)$$

where Φ is defined in section 11 and Q, \bar{Q} are defined in section 12. Therefore,

$$W_{tree} = \sum_{k=2}^{N_c} m_k U_k + \lambda Z \Rightarrow W_{tree,d} \equiv \left[\sum_{k=2}^{N_c} m_k \text{Tr} \Phi^k + \lambda Q \Phi \bar{Q} \right]_{\langle \Phi \rangle}, \quad (13.3)$$

¹⁸ The number of microscopic degrees of freedom (d.o.f(Φ, Q, \bar{Q})) minus the gauge freedom is $2N_c$, while the number of macroscopic degrees of freedom (d.o.f(U_k, X, Z)) is $N_c + 1$. This means that one might need to add the $N_c - 1$ gauge singlets $Z_k \equiv Q \Phi^k \bar{Q}$, $k = 2, \dots, N_c$ to the integrating in procedure. However, we checked that adding Z_k is irrelevant to the final result in the $SU(N_c)$ vacua branch (see also the footnote after eq. (13.15)).

namely, to find $W_{tree,d}$ we should solve the equation

$$\frac{\partial W_{tree}}{\partial \Phi^t} = \sum_{k=2}^{N_c} k m_k \Phi^{k-1} + \lambda Q \bar{Q} - \frac{1}{N_c} \left(\lambda X + \sum_{k=3}^{N_c} k m_k U_{k-1} \right) I = 0, \quad (13.4)$$

where I is the $N_c \times N_c$ identity matrix. The different solutions of eq. (13.4) correspond to different branches of classical vacua of the theory.

We are interested in a branch where Φ decouples as its mass approaches infinity: $m_2 \rightarrow \infty$. Therefore, in this limit, $W_i(m_2 \rightarrow \infty) = W_{tree,d} + W_\Delta \rightarrow 0$. Moreover, when $\Lambda \rightarrow 0$, $W_\Delta(\Lambda \rightarrow 0) \rightarrow 0$. Therefore, in the combined limit $m_2 \rightarrow \infty$ and $\Lambda \rightarrow 0$ both $W_\Delta \rightarrow 0$ and $W_i \rightarrow 0$, which implies that also $W_{tree,d} \rightarrow 0$. But $W_{tree,d}$ is independent of Λ and, therefore, we conclude that $W_{tree,d}(m_2 \rightarrow \infty) \rightarrow 0$. We refer to this branch as the “perturbative branch.”

In the Appendix, it is shown that requiring to have a branch with a discrete number of vacua, in addition to an appropriate behavior in the $\Lambda \rightarrow 0$ and $m_2 \rightarrow \infty$ limits, is consistent with a $W_{tree,d}$ evaluated at the single $\langle \Phi \rangle$ solution of eq. (13.4) which is perturbative in λ/m_2 . This solution reads

$$\Phi = \frac{\lambda}{2m_2} \left(\frac{X}{N_c} I - Q \bar{Q} \right) + \mathcal{O}((\lambda/m_2)^2). \quad (13.5)$$

Since

$$\text{Tr}_{N_c}(Q \bar{Q})^k = X^k, \quad (13.6)$$

the characteristic polynomial of the $N_c \times N_c$ matrix $Q \bar{Q}$ is

$$(Q \bar{Q})^{N_c-1} (Q \bar{Q} - X) = 0. \quad (13.7)$$

This implies that $Q \bar{Q}$ has an eigenvalue X and $N_c - 1$ zero eigenvalues. Therefore, to solve (13.4) we can choose a pair of bases for which

$$Q \bar{Q} = \text{diag}(0, \dots, 0, X). \quad (13.8)$$

In these bases, by using (13.5) one finds that the perturbative solution to eq. (13.4) reads

$$\Phi = \text{diag}(a, \dots, a, -(N_c - 1)a). \quad (13.9)$$

This is the solution which corresponds classically to the $SU(N_c - 1)$ vacua. Using (13.3), (13.4) and (13.9) we find that in this branch:

$$W_{tree,d} = - \sum_{k=2}^{N_c} (k-1)(N_c-1) \left[1 - (1-N_c)^{k-1} \right] m_k \langle a \rangle^k, \quad (13.10)$$

where $\langle a \rangle$ is the solution of

$$\lambda X - \sum_{k=2}^{N_c} k \left[1 - (1-N_c)^{k-1} \right] m_k a^{k-1} = 0, \quad (13.11)$$

for which $\langle a \rangle = \mathcal{O}(\lambda X/m_2)$ as $m_2 \rightarrow \infty$; there is a single solution obeying this condition.

In the Appendix, it is also shown that requiring to have a branch with a discrete number of vacua, in addition to an appropriate behavior in the $\Lambda \rightarrow 0$ and $m_2 \rightarrow \infty$ limits, implies that such a physical branch has $W_\Delta = 0$. Therefore, we find that in the $SU(N_c)$ vacua branch, the nonperturbative superpotential of the up theory is derived by

$$W_u = [W_d + W_{tree,d} - W_{tree}]_{\langle m_k \rangle, \langle \lambda \rangle}, \quad (13.12)$$

where W_d , $W_{tree,d}$ and W_{tree} are given in eqs. (13.1), (13.10) and (13.3), respectively. After some algebra one finds

$$W_{1,1}(U_2, X, Z) = -\Lambda^{-b_1} X \Gamma^{N_c} + \sum_{k=2}^{N_c} m_k U_k + mX + \lambda Z, \quad (13.13)$$

where recall $b_1 = 2N_c - 1$, and

$$\Gamma = U_2 - \frac{N_c}{N_c - 1} x^2, \quad x = ZX^{-1}, \quad (13.14)$$

and the $N_c - 2$ constraints:

$$U_k = \frac{(1 - N_c)^{k-1} - 1}{(1 - N_c)^{k-1}} x^k, \quad k = 3, \dots, N_c. \quad (13.15)$$

Equation (13.15) is a set of classical conditions (to check it, on the $SU(N_c - 1)$ classical vacua, one may use eqs. (13.8), (13.9)), while Γ in eq. (13.14)

vanishes classically, as expected physically due to the negative power of Λ in the superpotential (13.13). If one would ignore the “apparently irrelevant” operators U_k with $k > 2$, in the integrating in procedure, one would fail to get Γ which vanishes classically. In other words, one gets for Γ the characteristic polynomial for Φ (with Φ being replaced by x , up to an overall x -dependent factor); equation (13.14) is the value of the characteristic polynomial on the classical constraints in eq. (13.15), while ignoring the U_k with $k > 2$ means to set their values to zero in the characteristic polynomial, thus leading to an object that does not vanish classically.¹⁹

We now want to find the vacua of the theory in the branch (13.13), (13.15), namely, we should solve the equations of motion $\delta W_{1,1}/\delta U_k = \delta W_{1,1}/\delta X = \delta W_{1,1}/\delta Z = 0$ on the constraints (13.15). We study here only the case

$$m_k = 0, \quad k = 3, \dots, N_c. \quad (13.16)$$

The equations of motion read²⁰:

$$m_2 = N_c \Lambda^{-b_1} X \Gamma^{N_c-1}, \quad (13.17)$$

$$m = \Lambda^{-b_1} \Gamma^{N_c-1} \left(\Gamma - N_c x \frac{\partial \Gamma}{\partial x} \right), \quad (13.18)$$

$$\lambda = N_c \Lambda^{-b_1} \Gamma^{N_c-1} \frac{\partial \Gamma}{\partial x}. \quad (13.19)$$

Combining eq. (13.18) with eq. (13.19) we get

$$\Gamma^{N_c} = \Lambda^{b_1} \lambda (x + \mu), \quad \mu = \lambda^{-1} m. \quad (13.20)$$

Equations (13.20) and (13.19) are the singularity conditions of a genus $N_c - 1$ hyperelliptic curve defined by

$$y^2 = \Gamma(x)^{N_c} - \Lambda^{b_1} \lambda (x + \mu). \quad (13.21)$$

¹⁹ Unlike the U_k , the Z_k with $k > 2$, defined in the footnote before eq. (13.2), are indeed *irrelevant*. We checked that adding them to the integrating in procedure does not affect the result that $W_\Delta = 0$ and, consequently, does not change the final result in eqs. (13.13)-(13.15), but give extra (classical) constraints for Z_k : $Z_k X^{-1} = x^k$.

²⁰In the presence of tree-level terms with $m_k \neq 0$ for $k \geq 3$, the equations of motion receive m_k -dependent corrections, due to the constraints (13.15), and one gets a different vacua structure [20]: turning on $\sum_{k=3}^{N_c} m_k U_k$ give extra vacua not considered here.

Using eqs. (13.19), (13.20) to solve U_2 in terms of x , and eliminating U_2 in eq. (13.19) we find

$$x^{N_c}(x + \mu)^{N_c-1} - \left(\frac{1 - N_c}{2N_c^2}\right)^{N_c} \Lambda^{b_1} \lambda = 0, \quad (13.22)$$

and

$$U_2 = -\frac{N_c}{N_c - 1} x(b_1 x + 2N_c \mu). \quad (13.23)$$

Therefore, we find that $W_{1,1}$ (13.13) has $b_1 = 2N_c - 1 = N_c + (N_c - 1)N_f$ vacua, namely, the $2N_c - 1$ solutions for $M(x)$ in terms of the $2N_c - 1$ roots of the polynomial equation for x (13.22) – the singularities of the hyperelliptic curve (13.22) – and the solution for X given by eq. (13.17) (Z is now determined by $Z = xX$, and recall that $U_k, k > 2$ are fixed by (13.15)). These $b_1 = 2N_c - 1$ vacua are the vacua of the theory in the Higgs/confinement branch²¹. The phase transition points to the Coulomb branch are at $X = 0$. This happens iff the adjoint superfield is massless, namely

$$X = 0 \Leftrightarrow m_2 = 0. \quad (13.24)$$

The values of U_k at the $SU(N_c)$ vacua are independent of the value X . When $m = m_k = 0$, there is a \mathbf{Z}_{2N_c-1} transformation relating the different vacua; this is a symmetry of the U_k moduli space in the Coulomb phase.

14 $SU(2)$ with $N_A = N_f = 1$ revisited

In this section, we rederive the results of section 5 in a simpler way, similar to the manipulation for $N_c > 2$ in section 13.

²¹ Adding a tree-level superpotential $\sum_{k=3}^{N_c} m_k U_k$ gives rise, generically, to a total of $N_c^2 - 1$ solutions [20]; the extra $N_c(N_c - 2)$ solutions go to infinity in the U_k space when $m_k/m_2 \rightarrow 0$, $k \geq 3$, and one is left with the $2N_c - 1$ “ $SU(N_c)$ vacua” considered here. Moreover, adding a tree-level superpotential $\sum_{k=2}^{N_c} \lambda_k Z_k$ gives rise, generically, to a total of $2N_c(N_c - 1)$ solutions [20]; this is due to the constraints discussed in a footnote after eq. (13.15).

For $N_c = 2$, the anti-fundamental representation is equivalent to the fundamental representation and, therefore, Z of eq. (13.2) becomes a symmetric 2×2 matrix (see eq. (4.2)). Moreover, X of section 13 denotes $\text{Pf}X$ where X is an antisymmetric 2×2 matrix, and there is a single Casimir which we denote by $U_2 \equiv M$, as in section 5. We thus find that the superpotential $W_{1,1}$ is given by eq. (13.13) with (13.14) replaced by

$$\Gamma = M - 2x^2 = M + \text{Tr}(ZX^{-1})^2, \quad x = (\det Z)^{1/2}(\text{Pf}X)^{-1}. \quad (14.1)$$

Following the discussion in section 13, we find that the vacua are given by the solutions to eqs. (13.17), (13.18), and eq. (13.19) is modified to

$$(\det \lambda)^{1/2} = \Lambda^{-3} \Gamma \frac{\partial \Gamma}{\partial x}. \quad (14.2)$$

Namely, we find that X is solved by

$$\text{Pf}X = \frac{m_2 \Lambda^3}{2\Gamma} \quad (14.3)$$

($m_2 \equiv \tilde{m}$ in the notations of section 5), and x, M are given by the singularity conditions of an elliptic curve:

$$y^2 = \Gamma^2 - \Lambda^3(\alpha x + m), \quad \alpha \equiv 2(\det \lambda)^{1/2}. \quad (14.4)$$

The curve in the form (14.4) was presented in ref. [21]. This elliptic curve is related to the previous one, in eqs. (5.22), (5.23), by rescaling $m \rightarrow m/2$, $\lambda \rightarrow \lambda/\sqrt{2}$, together with an $SL(2, \mathbf{C})$ transformation:

$$x \rightarrow \frac{ax + b}{cx + d}, \quad y \rightarrow K(cx + d)^2 y, \quad ad - bc = 1, \quad (14.5)$$

where a, b, c, d and K are given in terms of $M, \alpha \Lambda^3$ and $m \Lambda^3$.

15 $SU(N_c)$ with $N_A, N_f < N_c$ ($b_1 = 3N_c - N_c N_A - N_f$)

In this section, we present the effective superpotential, W_{N_f, N_A} , in $N = 1$ supersymmetric $SU(N_c)$ gauge theory, $N_c > 2$, with N_A matter superfields

in the adjoint representation, Φ_α^{ab} , $\text{Tr}\Phi = 0$, and $N_f < N_c$ flavors, Q_i^a , $\bar{Q}_a^{\bar{i}}$ (when $N_f \geq N_c$ there are also baryons in the theory). Here $a, b = 1, \dots, N_c$, $i, \bar{i} = 1, \dots, N_f$, and $\alpha = 1, \dots, N_c$. As before, we derive the superpotential by integrating in N_A adjoint supermultiplets to a supersymmetric $SU(N_c)$ gauge theory with $N_f < N_c$ flavors, presented in section 12; the superpotential of the down theory is $W_d = W_{N_f,0}(X)$, given in eq. (12.2). We consider the up theories with one-loop asymptotic freedom or conformal invariance, for which

$$b_1 = 3N_c - N_f - N_c N_A \geq 0, \quad (15.1)$$

where $-b_1$ is the one-loop coefficient of the gauge coupling beta-function.

The relevant gauge singlets we should add to $X_i^{\bar{i}}$ in eq. (12.1) are

$$\begin{aligned} U_{(\alpha_1, \dots, \alpha_k)} &= \text{Tr}_{N_c}(\Phi_{\alpha_1} \cdots \Phi_{\alpha_k}), \quad k = 2, \dots, N_c, \quad \alpha_n = 1, \dots, N_A, \\ Z_{i\alpha}^{\bar{i}} &= \text{Tr}_{N_c}(Q_i \bar{Q}^{\bar{i}} \Phi_\alpha). \end{aligned} \quad (15.2)$$

(For $N_A = 1$, the gauge singlets $Z_{i(\alpha_1, \dots, \alpha_k)}^{\bar{i}} = \text{Tr}_{N_c}(Q_i \bar{Q}^{\bar{i}} \Phi_{\alpha_1} \cdots \Phi_{\alpha_k})$, $k = 2, \dots, N_c$, are irrelevant, as in the $N_f = 1$ case; they do not change the final result even if added to the integrating in procedure²². This is assumed also when $N_A = 2$.) We obtain the superpotential²³

$$\begin{aligned} W_{N_f, N_A}(X, U, Z) &= (b_1 - 2N_c) \left[\Lambda^{-b_1} \det_{N_f} X (\det_{N_A} \Gamma)^{N_c} \right]^{\frac{1}{2N_c - b_1}} \\ &+ \sum_{k=2}^{N_c} \left(\sum_{\alpha_1=1}^{N_A} \cdots \sum_{\alpha_k=1}^{N_A} m_{(\alpha_1, \dots, \alpha_k)} U_{(\alpha_1, \dots, \alpha_k)} \right) \\ &+ \text{Tr}_{N_f} m X + \text{Tr}_{N_f} \lambda^\alpha Z_\alpha, \end{aligned} \quad (15.3)$$

and the constraints

$$\begin{aligned} U_{(\alpha_1, \dots, \alpha_k)} &= \text{Tr}_{N_f}(Z_{\alpha_1} X^{-1} \cdots Z_{\alpha_k} X^{-1}) \\ &- \frac{1}{(N_f - N_c)^{k-1}} \text{Tr}_{N_f}(Z_{\alpha_1} X^{-1}) \cdots \text{Tr}_{N_f}(Z_{\alpha_k} X^{-1}), \end{aligned} \quad (15.4)$$

²² We did not discuss here the baryon-like operators of refs. [16, 22]; we checked that the operators, containing at most N_c adjoint superfields, are irrelevant for the integrating in procedure on the perturbative branch (although they might be important on the nonperturbative branches [20]).

²³ When $b_1 = 2N_c$, the nonperturbative superpotential vanishes and one obtains an additional constraint; this happens only in case $N_A = 1$, $N_f = 0$, considered in section 11.

where

$$\Gamma_{\alpha\beta} = U_{(\alpha,\beta)} - \text{Tr}_{N_f}(Z_\alpha X^{-1} Z_\beta X^{-1}) - \frac{1}{N_c - N_f} \text{Tr}_{N_f}(Z_\alpha X^{-1}) \text{Tr}_{N_f}(Z_\beta X^{-1}). \quad (15.5)$$

To get this result, for $N_A = 1$, we follow the strategy used in the $N_A = N_f = 1$ case in section 13. Namely, we use limiting considerations and impose the physical condition to have a finite number of $SU(N_c)$ vacua branches, to find that $W_\Delta = 0$. Then, using the perturbative branch, where $W_{tree,d}(m_{(\alpha,\beta)} \rightarrow \infty) \rightarrow 0$, and after some algebra, we find eqs. (15.3)-(15.5) for $N_A = 1$.

The $N_A = 3$, $N_f = 0$ case includes the $N = 4$ supersymmetric $SU(N_c)$ gauge theory. As for the $SU(2)$ case with $N_A = 3$, discussed in section 4, in this case, W must be equal to the Yukawa coupling tree-level term of the three adjoint superfields. This is, indeed, the result in eq. (15.3) for $N_A = 3$, $N_f = 0$. Moreover, by integrating out a single adjoint superfield, one obtains the result in (15.3) for $N_A = 2$, $N_f = 0$. The superpotential for $N_A = 2$, $N_f \neq 0$ in eq. (15.3) is conjectured. To get this result, we use the assumption that $W_\Delta = 0$ in the perturbative branch also in this case.

One may now find the quantum vacua of the theory, by solving the equations of motion derived from (15.3). In case there is a single adjoint matter superfield ($N_A = 1$, $N_f < N_c$), and setting

$$m_{(\alpha_1, \dots, \alpha_k)} = 0, \quad k > 2, \quad (15.6)$$

we find that the number of (branches of) $SU(N_c)$ vacua is

$$\begin{aligned} \text{no. of } SU(N_c) \text{ vacua} & \quad \text{for } N_f < N_c : \\ \frac{1}{2}b_1(N_f + 1) & = N_c + N_f(N_c - 1) - \frac{1}{2}N_f(N_f - 1). \end{aligned} \quad (15.7)$$

When, in addition, $m = m_{(\alpha_1, \alpha_2)} = 0$, there is a \mathbf{Z}_{b_1} symmetry relating the different vacua. This symmetry can be read directly from the quantum superpotential (15.3): it acts on Q , \bar{Q} and Φ by

$$\Phi \rightarrow e^{\frac{2\pi i n}{b_1}} \Phi, \quad Q \rightarrow e^{-\frac{\pi i n}{b_1}} Q, \quad \bar{Q} \rightarrow e^{-\frac{\pi i n}{b_1}} \bar{Q}, \quad n = 1, \dots, b_1, \quad (15.8)$$

and leave invariant both the tree-level term, $\text{Tr}_{N_f} \lambda^\alpha Z_\alpha$, and the nonperturbative superpotential, $(\Lambda^{-b_1} \det_{N_f} X (\det_{N_A} \Gamma)^{N_c})^{1/N_f}$.

What about the duality of refs. [16, 22]? It is valid when $N_c > 2$ and $N_f \geq N_c/(k-1)$, $k = 3, \dots, N_c$, depending on which $\text{Tr} \Phi^k$ interaction is turned on, and at the infra-red fixed point of the renormalization group flow. Yet, in the $SU(N_c)$ vacua branches, considered here, we do not have the tree-level couplings, $m_{(\alpha_1, \dots, \alpha_k)}$ with $k > 2$, which are required for the duality arguments of [16, 22]. Studying the cases where $m_{(\alpha_1, \dots, \alpha_k)} \neq 0$, as well as other branches of $SU(N_c)$ supersymmetric gauge theories, might be useful to understand this duality [20].

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Appendix A - $W_\Delta = 0$

In this Appendix, we show in detail the considerations leading to the conclusion that $W_\Delta = 0$ in the examples considered in sections 3,4,11,13.

A.1 - Down theory = $SU(2)$ with $N_A = 0$, $N_f \leq 4$, Up theory = $SU(2)$ with $N_A \neq 0$

The $U(1)_Q \times U(1)_\Phi \times U(1)_R$ quantum numbers of W are $(0, 0, 2)$ and, therefore,

$$W_\Delta(X, \Lambda, \tilde{m}, \lambda) = W_{tree,d} f(t), \quad (\text{A.1})$$

where $W_{tree,d}$ is given in eq. (4.3), and t denotes, schematically, any singlet of the $SU(2N_f)$ flavor symmetry with $(0, 0, 0)$ (Q, Φ, R) -charges. When inte-

grating in triplets to an $SU(2)$ theory with doublets, W_Δ depends on X, λ, Λ and \tilde{m} . The quantum numbers of $X, \lambda, \Lambda^{b_1}, \tilde{m}$ are

$$\begin{aligned} X &: (Q, \Phi, R) = (2, 0, 0), \\ \lambda &: (Q, \Phi, R) = (-2, -1, 2), \\ \Lambda^{b_1} &: (Q, \Phi, R) = (2N_f, 4N_A, 4 - 4N_A - 2N_f), \quad b_1 = 6 - 2N_A - N_f, \\ \tilde{m} &: (Q, \Phi, R) = (0, -2, 2). \end{aligned} \quad (\text{A.2})$$

Therefore, if we denote schematically,

$$t \sim (\Lambda^{b_1})^a \tilde{m}^b X^c \lambda^d, \quad (\text{A.3})$$

we find that the condition that t has $(0, 0, 0)$ (Q, Φ, R) -charges implies

$$(2 + 2N_A - N_f)a = b. \quad (\text{A.4})$$

We now want to impose the limits:

$$W_\Delta(\tilde{m} \rightarrow \infty) \rightarrow 0, \quad W_\Delta(\Lambda \rightarrow 0) \rightarrow 0. \quad (\text{A.5})$$

Therefore, we are only interested in the dependence of W_Δ on \tilde{m} and Λ . Recall that, by definition, $W_{tree,d}$ is Λ -independent. Using the schematic dependence of $W_{tree,d}$ (4.3) on \tilde{m}

$$W_{tree,d}(\tilde{m}) \sim \frac{1}{\tilde{m}}, \quad (\text{A.6})$$

and analyzing the several cases with $N_f \neq 0$, $b_1 \geq 0$ we find the following schematic \tilde{m}, Λ dependence:

- For $N_A = 1$, $N_f = 1, 2, 3$ ($b_1 = 4 - N_f$): equation (A.4) implies $b = b_1 a$ and, therefore,

$$W_\Delta(\tilde{m}, \Lambda) \sim \frac{1}{\tilde{m}} f((\tilde{m}\Lambda)^{b_1}). \quad (\text{A.7})$$

- For $N_A = 1$, $N_f = 4$ ($b_1 = 0$): “ Λ^{b_1} ” $\sim (\det \lambda)^{-1/2}$ (see eq. (8.13)) and, therefore, $t \sim \tilde{m}^b X^c \lambda^d$. The condition that t has $(0, 0, 0)$ (Q, Φ, R) -charges implies $b = c = d = 0$ and, therefore, t can only depend on τ_0 (introduced in section 8). We thus find

$$W_{(0,0,2)} \sim \frac{(\lambda X)^2}{\tilde{m}} f(\tau_0) \sim \frac{X^2}{\Lambda_d} f(\tau_0), \quad (\text{A.8})$$

where $\Lambda_d \equiv \Lambda_{N_f=4, N_A=0} \sim \tilde{m}(\det \lambda)^{-1/4}$ (see section 8). Using (A.8), holomorphy and $SU(8)$ flavor symmetry we find that

$$W_\Delta \sim W_{tree,d} f(\tau_0). \quad (\text{A.9})$$

The function $f(\tau_0)$ is related to the function $\beta(\tau_0)$ used in section 8 to rescale M ; here we can get rid of f in W_Δ by rescaling $\tilde{m} \rightarrow (1+f)\tilde{m}$ together with $M \rightarrow M/(1+f)$, which takes $W_{tree,d} + W_\Delta \rightarrow W_{tree,d}$ while leaving $\tilde{m}M$ invariant.

- For $N_A = 2$, $N_f = 1$ ($b_1 = 1$): equation (A.4) implies $b = 5a$ and, therefore,

$$W_\Delta \sim \frac{1}{\tilde{m}} f(\tilde{m}^5 \Lambda), \quad (\text{A.10})$$

- For $N_A = N_f = 2$ ($b_1 = 0$): the theory is infra-red free (see section 10).

Since we can trust the instanton expansion in the Higgs branch, we know that W_Δ depends on integer powers of Λ^{b_1} . Therefore, the limits (A.5) and eqs. (A.7), (A.9), (A.10) imply that for all infra-red nontrivial cases with $N_f \neq 0$, the intermediate superpotential W_i (2.6) behaves like $W_{tree,d}$. As we already included $W_{tree,d}$ in the procedure, we conclude that $W_\Delta = 0$ when integrating in N_A triplets. Finally, when $N_f = 0$, it is easy to show that $W_\Delta = 0$.

A.2 - Down theory = $SU(N_c)$ with $N_A = N_f = 0$, Up theory = $SU(N_c)$ with $N_A = 1$

From eq. (11.8) we obtain:

$$\begin{aligned} U_2 - (\Lambda^{b_1})^{1/N_c} &= \Omega^2 \partial_2 f, \\ U_k &= \Omega^k \partial_k f, \quad k = 3, \dots, N_c - 2, \\ U_{N_c-1} &= \Omega^{N_c-1} \left[N_c f - t_1 \partial_1 f - \sum_{k=2}^{N_c-2} (N_c - k) t_k \partial_k f \right], \\ U_{N_c} &= \Omega^{N_c} \left[-(N_c - 1) f + t_1 \partial_1 f + \sum_{k=2}^{N_c-2} (N_c - k - 1) t_k \partial_k f \right], \end{aligned} \quad (\text{A.11})$$

where

$$\Omega \equiv \frac{\Lambda}{t_1}, \quad \partial_k f \equiv \frac{\partial f}{\partial t_k} \quad (\text{A.12})$$

($f = f_{tree,d} + f_\Delta$ is given in eq. (11.9)). On the solution (A.11) we find that $W_u = 0$ for any f . The $N_c - 1$ equations for U_k , $k = 2, \dots, N_c$, in terms of the $N_c - 2$ variables t (see eqs. (11.9), (11.5)) define, in general, an $N_c - 2$ dimensional manifold of vacua. There will be a discrete set of vacua only if the U_k in eq. (A.11) turn out to be t -independent. This happens only for f which solves the equations:

$$\begin{aligned} \partial_k f &= C_k t_1^k, \quad k = 2, \dots, N_c - 2, \\ N_c f - t_1 \partial_1 f - \sum_{k=2}^{N_c-2} (N_c - k) t_k \partial_k f &= C_{N_c-1} t_1^{N_c-1}, \\ -(N_c - 1) f + t_1 \partial_1 f + \sum_{k=2}^{N_c-2} (N_c - k - 1) t_k \partial_k f &= C_{N_c} t_1^{N_c}, \end{aligned} \quad (\text{A.13})$$

where C_k are independent of t and Λ (the Λ -independence follows from $U(1)_\Phi \times U(1)_R$ charge conservation). For $N_c > 3$ ($N_c = 3$ will be considered separately), the general solution is

$$f = \sum_{k=2}^{N_c-2} C_k t_1^k t_k, \quad C_{N_c-1} = C_{N_c} = 0. \quad (\text{A.14})$$

In the limit $\Lambda \rightarrow 0$, the parameter t_1 defined in (11.5) goes to zero and, therefore, $f \rightarrow 0$. Since we impose $W_\Delta(\Lambda \rightarrow 0) \rightarrow 0$, we find $f_\Delta(\Lambda \rightarrow 0) \rightarrow 0$ and, therefore, $f_{tree,d}(\Lambda \rightarrow 0) \rightarrow 0$. But $f_{tree,d}(t_2, \dots, t_{N_c-2})$ is independent of Λ (because t_2, \dots, t_{N_c} , defined in (11.5), are Λ -independent) and, therefore

$$f_{tree,d} = 0. \quad (\text{A.15})$$

Equation (A.15) means that the condition to have a discrete set of vacua chooses the branch where $W_{tree,d} = 0$.²⁴

²⁴ It can be verified that this is the only branch of $W_{tree,d}$ which obeys: $W_{tree,d}(m_2 \rightarrow \infty) \rightarrow 0$.

For $N_c > 3$, the parameter τ , defined in (11.7), is m_2 independent. Therefore, eq. (11.10) and the condition $W_\Delta(m_2 \rightarrow \infty) \rightarrow 0$ imply $f_\Delta(m_2 \rightarrow \infty) \rightarrow 0$. In the limit $m_2 \rightarrow \infty$, we get from (11.5) that $t_2 \rightarrow \infty$ while $t_1, t_3, \dots, t_{N_c-2}$ are anything. Therefore, we conclude that $C_2 = \dots = C_{N_c-2} = 0$, which implies

$$f_\Delta = 0. \quad (\text{A.16})$$

We thus found that, for $N_c > 3$, $W_\Delta = W_{tree,d} = 0$, and from eq. (A.11) with $f = 0$ one finds eq. (11.12).

Finally, for $N_c = 3$, eqs. (11.7), (11.10) and (A.13) imply

$$f(t_1) = C_2 t_1^2 + C_3 t_1^3. \quad (\text{A.17})$$

As before, $W_\Delta(\Lambda \rightarrow 0) \rightarrow 0$ implies $W_{tree,d} = 0$ and, therefore, $W_\Delta = (C_2 t_1^2 + C_3 t_1^3) m_2^3 / m_3^2$. Now, the limit $W_\Delta(m_2 \rightarrow \infty) \rightarrow 0$ implies $C_2 = C_3 = 0$ and, therefore, $W_\Delta = 0$.

A.3 - Down theory = $SU(N_c)$ with $N_A = 0$, $N_f = 1$, Up theory = $SU(N_c)$ with $N_A = N_f = 1$

In the integrating in procedure (13.12) we involve $N_c + 2$ parameters: X, Λ, λ and m_k , $k = 2, \dots, N_c$. Therefore, we can construct $N_c - 1$ parameters with $(0, 0, 0)$ $U(1)_Q \times U(1)_\Phi \times U(1)_R$ quantum numbers:

$$\begin{aligned} t_0 &= \lambda \Lambda^{b_1} \left(\frac{m_{N_c}}{m_{N_c-1}} \right)^{b_1}, & t_1 &= \lambda X \frac{m_{N_c}^{N_c-2}}{m_{N_c-1}^{N_c-1}}, \\ t_k &= m_k \frac{m_{N_c}^{N_c-1-k}}{m_{N_c-1}^{N_c-k}}, & k &= 2, \dots, N_c - 2. \end{aligned} \quad (\text{A.18})$$

We also define the parameter τ with $(0, 0, 2)$ (Q, Φ, R) -charges

$$\tau \equiv \frac{m_{N_c}^{N_c}}{m_{N_c-1}^{N_c-1}}. \quad (\text{A.19})$$

Let us denote

$$f(t) \equiv f_d(t_0, t_1, t_2) + f_{tree,d}(t_1, \dots, t_{N_c-2}) + f_\Delta(t), \quad t \equiv (t_0, \dots, t_{N_c-2}), \quad (\text{A.20})$$

where $f_d, f_{tree,d}$ and f_Δ are defined by

$$W_d = \tau f_d(t_0, t_1, t_2), \quad W_{tree,d} = \tau f_{tree,d}(t_1, \dots, t_{N_c-2}), \quad W_\Delta = \tau f_\Delta(t). \quad (\text{A.21})$$

From eqs. (13.1), (A.18) we read:

$$f_d(t_0, t_1, t_2) = (N_c - 1) N_c^{\frac{-N_c}{N_c-1}} \left(\frac{t_0 t_2^{N_c}}{t_1} \right)^{\frac{1}{N_c-1}}. \quad (\text{A.22})$$

The function $f_{tree,d}$ can be a priori any branch of $W_{tree,d}$ in (13.3), (13.4). Since $W_\Delta(\Lambda \rightarrow 0) \rightarrow 0$, and as we trust the instanton expansion in the Higgs phase, f_Δ is holomorphic in t_0 and $f_\Delta(t_0 \rightarrow 0) \rightarrow 0$. Therefore,

$$f_\Delta(t) = \sum_{n=1}^{\infty} a_n(t_1, \dots, t_{N_c-2}) t_0^n, \quad (\text{A.23})$$

and, moreover, for $N_c > 3$ ($N_c = 3$ will be considered separately)

$$W_\Delta(m_2 \rightarrow \infty) \rightarrow 0 \Leftrightarrow f_\Delta(t_2 \rightarrow \infty) \rightarrow 0 \Leftrightarrow a_n(t_2 \rightarrow \infty) \rightarrow 0. \quad (\text{A.24})$$

Now, the integrating in procedure (13.12) reads:

$$W_u = \left[\tau f - \sum_{k=2}^{N_c} m_k U_k - \lambda Z \right]_{\langle m_k \rangle, \langle \lambda \rangle}, \quad (\text{A.25})$$

and we obtain

$$\begin{aligned} U_k &= \Omega^k \partial_k f, \quad k = 2, \dots, N_c - 2, \\ U_{N_c-1} &= \Omega^{N_c-1} \left[N_c f - b_1 t_0 \partial_0 f - \sum_{k=1}^{N_c-2} (N_c - k) t_k \partial_k f \right], \\ U_{N_c} &= \Omega^{N_c} \left[- (N_c - 1) f + b_1 t_0 \partial_0 f + \sum_{k=1}^{N_c-2} (N_c - 1 - k) t_k \partial_k f \right], \\ x &= \frac{\Omega}{t_1} [t_0 \partial_0 f + t_1 \partial_1 f], \quad x = Z X^{-1} \end{aligned} \quad (\text{A.26})$$

where

$$\Omega \equiv \left(\frac{\lambda \Lambda^{b_1}}{t_0} \right)^{1/b_1}, \quad \partial_k f \equiv \frac{\partial f}{\partial t_k}. \quad (\text{A.27})$$

Using (A.27) and the definitions, we get

$$W_u = \tau(-t_0 \partial_0 f)|_{\langle m_k \rangle, \langle \lambda \rangle} = -\Lambda^{-b_1} X x^{2N_c} B_1(t)|_{\langle m_k \rangle, \langle \lambda \rangle}, \quad (\text{A.28})$$

where

$$B_1(t) = \frac{t_0^2 t_1^{b_1} \partial_0 f}{(t_0 \partial_0 f + t_1 \partial_1 f)^{2N_c}}. \quad (\text{A.29})$$

Eliminating λ from eq. (A.26) we get

$$\frac{U_k}{x^k} = B_k(t), \quad k = 2, \dots, N_c, \quad (\text{A.30})$$

where

$$B_k(t) = \frac{t_1^k \partial_k f}{(t_0 \partial_0 f + t_1 \partial_1 f)^k}, \quad k = 2, \dots, N_c - 2, \quad (\text{A.31})$$

$$B_{N_c-1}(t) = \frac{t_1^{N_c-1} [N_c f - b_1 t_0 \partial_0 f - \sum_{k=1}^{N_c-2} (N_c - k) t_k \partial_k f]}{(t_0 \partial_0 f + t_1 \partial_1 f)^{N_c-1}}, \quad (\text{A.32})$$

$$B_{N_c}(t) = \frac{t_1^{N_c} [- (N_c - 1) f + b_1 t_0 \partial_0 f + \sum_{k=1}^{N_c-2} (N_c - 1 - k) t_k \partial_k f]}{(t_0 \partial_0 f + t_1 \partial_1 f)^{N_c}}. \quad (\text{A.33})$$

Equations (A.31), (A.32), (A.33) are $N_c - 1$ equations with $N_c - 1$ parameters $t \equiv (t_0, \dots, t_{N_c-2})$. So, in principle, we can solve t in terms of B_k , $k = 2, \dots, N_c$, and insert in (A.29), (A.28) to get W_u in terms of B_k , $k = 2, \dots, N_c$.

The equations of motion for $W = W_u + mX + \lambda Z$ with respect to variation of X and Z lead (after taking their combinations) to eqs. (13.20), (13.19) (with Γ^{N_c} being replaced by $x^{2N_c} B_1$), namely,

$$\begin{aligned} x^{2N_c} B_1 &= \Lambda^{b_1} (m + \lambda x), \\ \frac{\partial}{\partial x} (x^{2N_c} B_1) &= \Lambda^{b_1} \lambda. \end{aligned} \quad (\text{A.34})$$

Since B_1 depends on B_k with $k \geq 2$, these equations define, generically, a surface in the B_k space. This “surface” is a discrete set of points (vacua) iff B_1 depends on one B_k . Without loss of generality, we can express all B_k ’s in terms of B_2 :

$$\text{discrete set of vacua} \Leftrightarrow B_1 = F_1(B_2), \quad B_k = F_k(B_2), \quad k = 3, \dots, N_c. \quad (\text{A.35})$$

We can now use the properties of the $\Lambda \rightarrow 0$ and $m_2 \rightarrow \infty$ limits to show that a discrete set of vacua is obtained iff when $W_{tree,d}$ is in the λ/m_2 perturbative branch (see section 13) then $W_\Delta = 0$.

In the limit $\Lambda \rightarrow 0$, *i.e.*, $t_0 \rightarrow 0$ (recall eq. (A.18)), we impose $W_\Delta(\Lambda \rightarrow 0) \rightarrow 0$ and, therefore, $f_\Delta(t_0 \rightarrow 0) \rightarrow 0$. Moreover, (A.22) implies that $f_d(t_0 \rightarrow 0) \rightarrow 0$. We denote

$$B_k^{(0)} \equiv B_k(t_0 = 0, t_1, \dots, t_{N_c-2}), \quad (\text{A.36})$$

and for $t_0 = 0$ eqs. (A.29), (A.31), (A.32) and (A.33) read:

$$B_1^{(0)} = 0, \quad (\text{A.37})$$

$$B_k^{(0)} = \frac{\partial_k f_{tree,d}}{(\partial_1 f_{tree,d})^k}, \quad k = 2, \dots, N_c - 2, \quad (\text{A.38})$$

$$B_{N_c-1}^{(0)} = \frac{N_c f_{tree,d} - \sum_{k=1}^{N_c-2} (N_c - k) t_k \partial_k f_{tree,d}}{(\partial_1 f_{tree,d})^{N_c-1}}, \quad (\text{A.39})$$

$$B_{N_c}^{(0)} = \frac{-(N_c - 1) f_{tree,d} + \sum_{k=1}^{N_c-2} (N_c - 1 - k) t_k \partial_k f_{tree,d}}{(\partial_1 f_{tree,d})^{N_c}}. \quad (\text{A.40})$$

Since eqs. (A.37), (A.35) imply $B_1^{(0)} = F_1(B_2^{(0)}) = 0$, it follows that $B_2^{(0)}$ must be a constant in (t_1, \dots, t_{N_c-2}) (otherwise we would have $F_1(B_2) = 0$ for any B_2 which implies the unphysical result: $W_u = 0$ in eq. (A.28)). Now, eq. (A.35) implies $B_k^{(0)} = F_k(B_2^{(0)})$ and, therefore, we conclude that $B_k^{(0)}$, $k = 2, \dots, N_c$ are constants.

There is only a finite number of branches of $W_{tree,d}$. Therefore, to find which $f_{tree,d}$ has all $B_k^{(0)} = \text{constant}$, we insert $\partial_k f_{tree,d}$ from eq. (A.38) in eqs. (A.39), (A.40) and, after taking their combination, we obtain

$$t_1 + \sum_{k=2}^{N_c} k t_k B_k^{(0)} (\partial_1 f_{tree,d})^{k-1} = 0, \quad (\text{A.41})$$

$$f_{tree,d} = t_1 \partial_1 f_{tree,d} + \sum_{k=2}^{N_c} t_k B_k^{(0)} (\partial_1 f_{tree,d})^k, \quad (\text{A.42})$$

where $t_{N_c-1} = t_{N_c} = 1$ by convention. Equation (A.41) implies that

$$\zeta \equiv \partial_1 f_{tree,d} = -\frac{t_1}{2B_2^{(0)}t_2} + \mathcal{O}(t_2^{-2}), \quad (\text{A.43})$$

and thus $W_{tree,d}$ is in the perturbative branch in $1/m_2$, namely, in $1/t_2$ (recall eq. (A.18)), since (A.42), (A.43) give ²⁵

$$f_{tree,d} = -\frac{t_1^2}{4B_2^{(0)}t_2} + \mathcal{O}(t_2^{-2}). \quad (\text{A.44})$$

The perturbative branch of $W_{tree,d}$ is unique (see section 13). In this branch, from eqs. (13.10), (13.11) one finds

$$\lambda X + \sum_{k=2}^{N_c} km_k \frac{(1 - N_c)^{k-1} - 1}{(1 - N_c)^{k-1}} \left(\frac{\partial W_{tree,d}}{\partial(\lambda X)} \right)^{k-1} = 0. \quad (\text{A.45})$$

Using (A.18), (A.41), (A.42), (A.43) and (A.45) we conclude that

$$\sum_{k=2}^{N_c} kt_k \left(B_k^{(0)} + \frac{1 - (1 - N_c)^{k-1}}{(1 - N_c)^{k-1}} \right) \zeta^{k-1} = 0, \quad (\text{A.46})$$

for any t , which implies

$$B_k^{(0)} = \frac{(1 - N_c)^{k-1} - 1}{(1 - N_c)^{k-1}}. \quad (\text{A.47})$$

We now assume that $f_\Delta \neq 0$, and denote by $a_n(t_1, \dots, t_{N_c-2})t_0^n$ the lowest non-zero order term of (A.23); we will show that the $m_2 \rightarrow \infty$ limit implies $a_n = 0$ and, therefore, $f_\Delta = 0$. Recall that $B_k^{(0)}$ are the zero order terms of the t_0 expansion of $B_k(t)$. To next order in t_0 we obtain

$$B_1(t) = \left(\frac{N_c}{N_c - 1} \frac{f_d}{t_2 \zeta^2} \right)^{N_c} + \frac{t_0^{n+1}}{t_1 \zeta^{2N_c}} \left[na_n - \frac{2N_c}{N_c - 1} f_d \frac{na_n + t_1 \partial_1 a_n}{t_1 \zeta} \right] + \dots, \quad (\text{A.48})$$

²⁵ Actually, there are $N_c - 1$ solutions to eq. (A.41); one of them is the perturbative solution. As explained in section 13, this is the physical branch where the adjoint matter decouples in the infinite mass limit ($m_2 \rightarrow \infty$). We also expect to have other physical branches, where the adjoint matter decouples when $m_k \rightarrow \infty$ for $k > 2$. Nevertheless, all $N_c - 1$ solutions to eq. (A.41) give rise to the same W_u by integrating in.

$$\begin{aligned}
B_2(t) = B_2^{(0)} &+ \left(\frac{N_c}{N_c - 1} \frac{f_d}{t_2 \zeta^2} \right) \\
&+ \frac{t_0^n}{t_2 \zeta^2} \left[t_2 \partial_2 a_n - 2 \left(\frac{n a_n + t_1 \partial_1 a_n}{t_1 \zeta} \right) \left(\frac{N_c}{N_c - 1} f_d + t_2 \partial_2 f_{tree,d} \right) \right] \\
&+ \dots,
\end{aligned} \tag{A.49}$$

$$B_k(t) = B_k^{(0)} + \frac{t_0^n}{\zeta^k} \left[\partial_k a_n - \frac{k B_k^{(0)} \zeta^{k-1}}{t_1} (n a_n + t_1 \partial_1 a_n) \right] + \dots, \quad k = 3, \dots, N_c - 2, \tag{A.50}$$

$$\begin{aligned}
B_{N_c-1}(t) = B_{N_c-1}^{(0)} &+ \frac{t_0^n}{\zeta^{N_c-1}} \left[- (N_c - 1) B_{N_c-1}^{(0)} \frac{\zeta^{N_c-2}}{t_1} (n a_n + t_1 \partial_1 a_n) \right. \\
&+ \left. (N_c - n b_1) a_n - \sum_{k=1}^{N_c-2} (N_c - k) t_k \partial_k a_n \right] + \dots,
\end{aligned} \tag{A.51}$$

$$\begin{aligned}
B_{N_c}(t) = B_{N_c}^{(0)} &+ \frac{t_0^n}{\zeta^{N_c}} \left[- N_c B_{N_c}^{(0)} \frac{\zeta^{N_c-1}}{t_1} (n a_n + t_1 \partial_1 a_n) \right. \\
&- \left. (N_c - 1 - n b_1) a_n + \sum_{k=1}^{N_c-2} (N_c - 1 - k) t_k \partial_k a_n \right] \\
&+ \dots,
\end{aligned} \tag{A.52}$$

where “...” mean higher orders in t_0 ,²⁶ and ζ is given in (A.43).

Using eqs. (A.22), (A.35) and (A.48)-(A.52), we find that the $a_n(t_1, \dots, t_{N_c-2})$ must satisfy:

$$t_2 \partial_2 a_n - n a_n - 2 B_2^{(0)} \frac{t_2}{t_1} \zeta (n a_n + t_1 \partial_1 a_n) = C_2 \frac{t_2^{n+1}}{t_1^n} \zeta^{2N_c-2(N_c-1)(n+1)}, \tag{A.53}$$

$$\partial_k a_n - k B_k^{(0)} \frac{\zeta^{k-1}}{t_1} (n a_n + t_1 \partial_1 a_n) = C_k \left(\frac{t_2}{t_1} \right)^n \zeta^{k-2n(N_c-1)}, \quad k = 3, \dots, N_c - 2, \tag{A.54}$$

$$\begin{aligned}
(N_c - n b_1) a_n &- \sum_{k=1}^{N_c-2} (N_c - k) t_k \partial_k a_n - (N_c - 1) B_{N_c-1}^{(0)} (n a_n + t_1 \partial_1 a_n) \frac{\zeta^{N_c-2}}{t_1} \\
&= C_{N_c-1} \left(\frac{t_2}{t_1} \right)^n \zeta^{(1-2n)(N_c-1)},
\end{aligned} \tag{A.55}$$

²⁶ Recall that f_d , appearing in B_1, B_2 , depends on t_0 (see eq. (A.22)).

$$\begin{aligned}
(1 - N_c + nb_1)a_n &+ \sum_{k=1}^{N_c-2} (N_c - 1 - k)t_k \partial_k a_n - N_c B_{N_c}^{(0)}(na_n + t_1 \partial_1 a_n) \frac{\zeta^{N_c-1}}{t_1} \\
&= C_{N_c} \left(\frac{t_2}{t_1} \right)^n \zeta^{N_c-2n(N_c-1)},
\end{aligned} \tag{A.56}$$

where the constants C_k , $k = 2, \dots, N_c$ are defined as follows. C_2 is proportional to the next to leading order coefficient in the expansion of B_1 in powers of $B_2 - B_2^{(0)}$ (recall eq. (A.35)):

$$B_1 - (B_2 - B_2^{(0)})^{N_c} \sim C_2 (B_2 - B_2^{(0)})^{(N_c-1)(n+1)} + \dots \tag{A.57}$$

C_k , $k = 3, \dots, N_c$ are proportional to the coefficients of the leading order terms in the expansion of $B_k - B_k^{(0)}$ in powers of $B_2 - B_2^{(0)}$ (recall eq. (A.35)):

$$B_k - B_k^{(0)} \sim C_k (B_2 - B_2^{(0)})^{(N_c-1)n} + \dots, \quad k = 3, \dots, N_c. \tag{A.58}$$

The solution of eqs. (A.53)-(A.56) is

$$a_n = \left(\frac{t_2}{t_1 \zeta^{2(N_c-1)}} \right)^n \sum_{k=2}^{N_c} C_k \zeta^k t_k. \tag{A.59}$$

It is now time to use the limit $m_2 \rightarrow \infty$, namely, $t_2 \rightarrow \infty$ (recall eq. (A.18)) to show that $a_n = 0$. In this limit $f_\Delta(t_2 \rightarrow \infty) \rightarrow 0$ and, therefore, $a_n(t_2 \rightarrow \infty) \rightarrow 0$. Recall the perturbative behavior of $\zeta(1/t_2)$ in eq. (A.43), we find that eq. (A.53) implies $C_2 = 0$, eq. (A.54) implies $C_k = 0$, $k = 3, \dots, N_c - 2$, eq. (A.55) implies $C_{N_c-1} = 0$, and eq. (A.56) implies $C_{N_c} = 0$; to summarize:

$$C_k = 0, \quad k = 2, \dots, N_c. \tag{A.60}$$

Inserting (A.60) in eq. (A.59) we find

$$a_n = 0. \tag{A.61}$$

Therefore, $f_\Delta = 0$ (recall eq. (A.23)) and we conclude that, for $N_c > 3$, $W_\Delta = 0$ on the $SU(N_c)$ vacua branch where $W_{tree,d}$ is the one perturbative in $1/m_2$.

We now consider the $N_c = 3$ case. For $SU(3)$, eq. (13.3) reads:

$$W_{tree,d} \equiv \left[m_2 \text{Tr} \Phi^2 + m_3 \text{Tr} \Phi^3 + \lambda Q \Phi \bar{Q} \right]_{\langle \Phi \rangle}. \quad (\text{A.62})$$

We find $W_{tree,d} = \tau f_{tree,d}(t_1)$, where τ and t_1 are given by eqs. (A.19) and (A.18), respectively, with $N_c = 3$,²⁷ and in the nonperturbative branch

$$f_{tree,d}(t_1) = \frac{8}{9} + \frac{2}{3}t_1, \quad (\text{A.63})$$

while in the perturbative branch

$$f_{tree,d}(t_1) = \frac{4}{9} \left[1 - \frac{3}{2}t_1 \pm (1 - t_1)^{3/2} \right]. \quad (\text{A.64})$$

Indeed, in the perturbative branch²⁸, $W_{tree,d} \rightarrow 0$ when $m_2 \rightarrow \infty$. We now follow eqs. (A.20)-(A.23). Since $W_\Delta(\Lambda \rightarrow 0) \rightarrow 0$, and as we trust the instanton expansion in the Higgs phase, f_Δ is holomorphic in t_0 , defined in eq. (A.18) with $N_c = 3$, $b_1 = 5$, and imposing also $W_\Delta(m_2 \rightarrow \infty) \rightarrow 0$ we find

$$f_\Delta(t_0, t_1) = \sum_{n=1}^{\infty} a_n(t_1) t_0^n, \quad (\text{A.65})$$

$$a_n(t_1) t_1^{(5n-3)/2} (t_1 \rightarrow 0) \rightarrow 0. \quad (\text{A.66})$$

Following eqs. (A.25)-(A.47), with $N_c = 3$, we find that imposing the $\Lambda \rightarrow 0$ behavior and a discrete set of vacua give rise to the unphysical result $W_u = 0$ in the nonperturbative branch (A.63), while for the perturbative branch ((A.64) with the minus sign), assuming $f_\Delta \neq 0$ and denoting by $a_n(t_1) t_0^n$ the lowest order non-zero term in (A.65), we obtain that eqs. (A.48)-(A.52) are being replaced with

$$B_1 = \left(\frac{3}{2} \frac{f_d}{\zeta^2} \right)^3 + t_0^{n+1} \left(\frac{n a_n}{t_1 \zeta^6} \right) + \dots, \quad (\text{A.67})$$

²⁷Note that, unlike $N_c \neq 3$, here τ and t_0, t_1 depend on m_2 .

²⁸ We consider both \pm possibilities (the $N_c - 1 = 2$ solutions of eq. (A.41) with $N_c = 3$) as the “perturbative branch” because they give rise to the same W_u (see the discussion for $N_c > 3$); so we may choose to work with the one obeying $W_{tree,d}(m_2 \rightarrow \infty) \rightarrow 0$.

$$B_2 = \frac{3}{2} + \left(\frac{3}{2}\frac{f_d}{\zeta^2}\right) + \frac{t_0^n}{\zeta^2} \left[(3-5n)a_n - 2t_1a'_n - \frac{3\zeta}{t_1}(na_n + t_1a'_n) \right] + \dots, \quad (\text{A.68})$$

$$B_3 = \frac{3}{4} + \frac{t_0^n}{\zeta^3} \left[(5n-2)a_n + t_1a'_n - \frac{9\zeta^2}{4t_1}(na_n + t_1a'_n) \right] + \dots \quad (\text{A.69})$$

We thus find that the expansion of B_1 and B_3 in terms of $B_2 - B_2^{(0)} = B_2 - 3/2$ is

$$\begin{aligned} B_1 &= \left(B_2 - \frac{3}{2}\right)^3 + \left(B_2 - \frac{3}{2}\right)^{2(n+1)} 3^{n+1} \zeta^{4n-2} t_1^n \left[(6n-3)a_n + 2t_1a'_n \right. \\ &\quad \left. + \frac{3\zeta}{t_1}(na_n + t_1a'_n) \right] + \dots, \end{aligned} \quad (\text{A.70})$$

$$\begin{aligned} B_2 &= \frac{3}{4} + \left(B_2 - \frac{3}{2}\right)^{2n} 3^n \zeta^{4n-3} t_1^n \left[(5n-2)a_n + t_1a'_n - \frac{9\zeta^2}{4t_1}(na_n + t_1a'_n) \right] \\ &\quad + \dots \end{aligned} \quad (\text{A.71})$$

Now, the condition (A.35) implies

$$(3-6n)a_n - 2t_1a'_n - \frac{3\zeta}{t_1}(na_n + t_1a'_n) = C_2 \frac{\zeta^{2-4n}}{t_1^n}, \quad (\text{A.72})$$

$$(5n-2)a_n + t_1a'_n - \frac{9\zeta^2}{4t_1}(na_n + t_1a'_n) = C_3 \frac{\zeta^{3-4n}}{t_1^n}. \quad (\text{A.73})$$

The solution of eqs. (A.72), (A.73) is

$$a_n = \frac{1}{(t_1\zeta^4)^n} (C_2\zeta^2 + C_3\zeta^3), \quad (\text{A.74})$$

where ζ is given in (A.43) with $t_2 = 1$. So finally, we got in eq. (A.74) the result (A.59) with $N_c = 3$ and $t_2 = t_3 = 1$.

Finally, using the $m_2 \rightarrow \infty$ limit which implies (A.66), we find that $C_2 = C_3 = 0$, and eq. (A.74) implies $a_n = 0$. Therefore, $f_\Delta = 0$, and also for $N_c = 3$ on the $SU(3)$ vacua branch, where $W_{tree,d}$ is the one perturbative in $1/m_2$, we conclude that $W_\Delta = 0$.

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